THE THEORY OF FUNCTIONS OF REAL VARIABLES

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VOLUME I

JAMES PIERPONT

LECTURES

ON

THE THEORY OF FUNCTIONS OF REAL VARIABLES

VOLUME I

BY

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PREFACE

The present work is based on lectures which the author is accustomed to give at Yale University on advanced calculus and the theory of functions of real variables. It falls in two volumes, and the following remarks apply only to the first.

The student of mathematics, on entering the graduate school of American universities, often has no inconsiderable knowledge of the methods and processes of the calculus. He knows how to differentiate and integrate complicated expressions, to evaluate indeterminate forms, to find maxima and minima, to differentiate a definite integral with respect to a parameter, etc. But no emphasis has been placed on the conditions under which these processes are valid. Great is his surprise to learn that they do not always lead to correct results. Numerous simple examples, however, readily convince him that such is nevertheless the case.

The problem therefore arises to examine more carefully the conditions under which the theorems and processes of the calculus are correct, and to extend as far as possible or useful the limits of their applicability.

In doing this it soon becomes manifest that the style of reasoning which the student has heretofore employed must be abandoned. Examples of curves without tangents, of curves completely filling areas, and other strange configurations so familiar to the analyst of to-day, make it clear that the rough and ready reasoning which rests on geometric intuition must give way to a finer and more delicate analysis. It is necessary for him to learn to think in the ϵ, δ forms of Cauchy and Weierstrass.

We have here the beginnings of the theory of functions of real variables, and the twofold problem just sketched characterizes sufficiently well the subject-matter and form of treatment of the present volume.

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To obtain a foundation, the author has begun by developing the real number system after the manner of Cantor and Dedekind, postulating the theory of positive integers. To obtain sufficient generality, he has employed from the start the more simple properties of point aggregates. No attempt, however, has been made to state every theorem with all possible generality. The author has allowed himself a wide liberty in this respect. Some theorems are stated under very broad conditions, while others are enunciated under extremely narrow ones. Some of these latter will be taken up later on.

Two features of this volume may be mentioned here. In the first place, the Euclidean form of exposition has been adopted. Each theorem with its appropriate conditions is stated and then proved. Without doubt this makes the book less attractive to read, but on the other hand it increases its usefulness as a book of reference. One is thus often saved the labor of running through a complicated piece of reasoning to pick up sundry conditions which have been introduced, sometimes without any explicit mention, in the course of the demonstration.

Secondly, numerous examples of incorrect forms of reasoning currently found in standard works on the calculus have been scattered through the earlier part of the volume. It is the author's experience that nothing stimulates the student's critical sense so powerfully as to ask him to detect the flaws in a piece of reasoning which at an earlier stage of his training he considered correct.

A few new terms and symbols have been introduced, but only after long deliberation. It is hoped that their employment sufficiently facilitates the reasoning, and the enunciation of certain theorems, to justify their introduction. It may be well to note here the author's use of the word "any" in the sense of *any* one at pleasure, and not in the sense of *some* one. The words "each," "every," "some," "any," are often used in an indiscriminate manner, and to this is due a part of the difficulty the beginner experiences in modern rigorous analysis.

No attempt has been made to attribute the various results here given to their respective authors. That has been rendered unnecessary by the very full bibliographies of the *Encyclopädie der*

Mathematischen Wissenschaften. The author feels it his pleasant duty, however, to acknowledge his large indebtedness to the writings of Jordan, Stolz, and Vallée-Poussin. He hopes, however, that it will be found that he has not used them servilely, but in an individual and independent manner.

Finally, he wishes to express his hearty thanks to his friend Professor M. B. Porter, and to his former pupil Dr. E. L. Dodd, for the unflagging interest they have shown during the composition of this volume and for their many and valuable suggestions.

JAMES PIERPONT.

NEW HAVEN, CONN., August, 1905.

NOTE

A list of some of the mathematical *terms* and *symbols* employed in this work will be found at the end of the volume.

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FUNCTION THEORY OF REAL VARIABLES

CHAPTER I

RATIONAL NUMBERS

Historical Introduction

1. The reader is familiar with the classification of real numbers into rational and irrational numbers. The rational numbers are subdivided into integers and fractions.

Besides the real numbers there is another class of numbers currently employed in modern analysis, viz. complex or imaginary numbers. In this work we shall deal almost exclusively with real numbers.

Historically, the first numbers to be considered were the positive integers 1, 2, 3, 4, 5, 6, . . . $(\mathfrak{J}$

We shall denote this system of numbers by $\mathfrak{J}$.

It is not our intention to develop the theory of these numbers; instead, we shall merely call attention to some of their fundamental properties.*

In the first place, we observe that the elements of $\mathfrak{J}$ are arranged in a certain fixed order; that is, if a, b are two *different* numbers, then one of them, say a, precedes the other b. This we express by saying that a is *less than* b, or that b is *greater than* a. In symbols

$$a < b, \qquad b > a.$$

---

* For an extended treatment of this subject we refer to the excellent work of O Stolz and J. A. Gmeiner, *Theoretische Arithmetik*, Leipzig, 1900.

We say the system $\mathfrak{J}$ is *ordered*. Furthermore, if $a=b$, $b=c$, then $a=c$. Also if $a=b$ and $b>c$, then $a>c$. Also if $a>b$, $b>c$, then $a>c$. Secondly, we observe that the system $\mathfrak{J}$ is infinite; after each element a follows another element, and so on without end.

On the elements of $\mathfrak{J}$ we perform four operations, viz. addition, subtraction, multiplication, and division. They are called *the four rational operations*. Of these operations, two may be regarded as direct, viz. addition and multiplication. The other two are their inverses, viz. subtraction, the inverse of addition; and division, the inverse of multiplication.

The formal laws governing addition are: the *associative law*, expressed by the formula

$$a+(b+c)=(a+b)+c;$$

and the *commutative law*, expressed by

$$a+b=b+a.$$

As regards the position of $a+b$ in the system $\mathfrak{J}$, relative to a or b, we have

$$a+b>a \text{ or } b.$$

We have also the relation

$$a+b>a'+b, \text{ if } a>a'.$$

The formal laws governing multiplication are the three following:

The *associative law*, expressed by

$$a\cdot bc=ab\cdot c.$$

The *distributive law*, expressed by

$$(a+b)c=ac+bc, \qquad a(b+c)=ab+ac.$$

The *commutative law*, expressed by

$$ab=ba.$$

We have also the relation, with respect to order,

$$ab>a'b \text{ if } a>a'.$$

Another important property is this:

If $$ac = bc, \text{ then } a = b.$$

The result of subtracting b from a is defined to be the number x in $\mathfrak{J}$, satisfying the relation

$$a = b + x.$$

But when $a \leqq b$, no such number exists in $\mathfrak{J}$.

Similarly, the result of dividing a by b is defined to be the number x in $\mathfrak{J}$, satisfying the relation

$$a = bx.$$

If, however, a is not a multiple of b, no such number exists in $\mathfrak{J}$.

Thus when we limit ourselves to the number system $\mathfrak{J}$, the two operations of subtraction and division cannot always be performed. In order that they may be, we enlarge our number system by introducing new elements, viz. fractions and negative numbers.

The introduction of fractions into arithmetic was comparatively easy; on the contrary, the negative numbers caused a great deal of trouble. For a time negative numbers were called absurd or fictitious. That the product of two of these fictitious numbers, $-a$ and $-b$, could give a real number, $+ab$, was long a stumbling block for many good minds.

The introduction of irrational numbers, *i.e.* numbers like

$$\sqrt{2}, \quad \sqrt[3]{5}, \quad \pi = 3.14159 \cdots, \quad e = 2.7182 \cdots,$$

never excited much comment. In actual calculations one used approximate rational values, and it was perfectly natural to subject them to the same laws as rational numbers. It is true that the Greeks of the time of Euclid were perfectly aware of the difficulties which beset a rigorous theory of incommensurable magnitudes; witness the fifth and tenth book of Euclid's *Elements*. But these subtle speculations found little attention during the Renaissance of mathematics in the seventeenth and eighteenth centuries. The contemporaries and successors of Newton and Leibnitz were too much absorbed in developing and applying the infinitesimal calculus to think much about its foundations.

At the close of the eighteenth and the beginning of the nineteenth centuries a change of attitude is observed. Gauss, Lagrange, Cauchy, and Abel called for a return to the rigor of the ancient Greek geometers. Certain paradoxes and even results obviously false had been obtained by methods in good repute. It became evident that the foundations of the calculus required a critical revision.

Abel in a letter to Hansteen in 1826 writes:* "I mean to devote all my strength to spread light in the immense obscurity which prevails to-day in analysis. It is so devoid of all plan and system that one may well be astonished that so many occupy themselves with it, — what is worse, it is absolutely devoid of rigor. In the higher analysis there exist very few propositions which have been demonstrated with complete rigor. Everywhere one observes the unfortunate habit of generalizing, without demonstration, from special cases; it is indeed marvelous that such methods lead so rarely to so-called paradoxes."

In another place he writes:† "I believe you could show me but few theorems in infinite series to whose demonstration I could not urge well-founded objections. The binomial theorem itself has never been rigorously demonstrated. . . . Taylor's expansion, the foundation of the whole calculus, has not fared better."

The critical movement inaugurated by the above-mentioned mathematicians found its greatest exponent in Weierstrass. It is no doubt largely due to his teachings that we may boast to-day that the great structure of modern analysis is built on the securest foundations known; that its methods have attained, if not surpassed, the justly famed rigor of the ancient Greek geometers. The saying of D'Alembert, "Allez en avant, la foi vous viendra," has lost its force. To-day, it is not faith that is required, but a little patience and maturity of mind.

As Weierstrass has shown, it is necessary, in order to place analysis on a satisfactory basis, to go to the very root of the matter and create a theory of irrational numbers with the same care and rigor as contemplated by Euclid, in his theory of incommensurable magnitudes, only on a far grander scale. It is too early to make the reader see the necessity of this step, but it will appear over and over again in the course of this work.

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* Abel, *Œuvres*, 2° ed., Vol. 2, p. 263.

† Abel, *l.c.*, p. 257.

Fractions

2. Before taking up the theory of irrational numbers, we wish to develop in some detail the modern theory of fractions and negative numbers. We shall rest our treatment of these numbers on the properties of the positive integers $\mathfrak{I}$, which we therefore suppose given. One of these properties, on account of its importance, deserves especial mention, viz.:

If the product ab is divisible by c, and if a and c are relatively prime, then b is divisible by c.

3. Let us begin with the positive fractions. As we saw, division of a by b, where a, b are two numbers in $\mathfrak{I}$, is not possible unless a is a multiple of b. Our object is therefore to form a new system of numbers, call it $\mathfrak{F}$, formed of the numbers of $\mathfrak{I}$ and certain other numbers, in which division shall be always possible.

We start by forming all possible pairs of numbers in $\mathfrak{I}$. These pairs we represent by the notation

$$\alpha = (a,\ a'),\quad \beta = (b,\ b') \cdots$$

In any one of these pairs, as $\alpha = (a,\ a')$, we call a the *first constituent* and a' the *second constituent* of α.

The system $\mathfrak{F}$ consists of the totality of these pairs α, β, $\cdots$

The elements of $\mathfrak{F}$ we have represented by the symbol $(a,\ a')$. Any other symbol would do. The customary ones are a/b and $a:b$.

We have purposely avoided these symbols, so familiar to the reader, in order that his attention shall be more closely fixed on the logical processes employed.

4. The objects of $\mathfrak{F}$ have as yet no properties; we proceed to assign them one arithmetic property after another, taking care that no property shall contradict preceding ones. We begin by setting $\mathfrak{F}$ in relation to $\mathfrak{I}$. We say: $(a,\ a')$ shall be a number c, in $\mathfrak{I}$, when $a = a'c$. Thus, *any element of $\mathfrak{F}$ whose first constituent is a multiple of its second, is an element of $\mathfrak{I}$*, i.e. *a positive integer.*

From this follows that every number a of $\mathfrak{J}$ lies in $\mathfrak{F}$. For, $(a, 1)$ lies in $\mathfrak{F}$. On the other hand

$$(a, 1) = a.$$

Hence a lies in $\mathfrak{F}$.

5. We define next the terms, *equal*, *greater than*, *less than*.

Let $$\alpha = (a, a'), \quad \beta = (b, b').$$

We say: $\alpha \gtreqless \beta$ according as $ab' \gtreqless a'b$.

We observe that to decide the equality or inequality of two elements in $\mathfrak{F}$, the operations required are on the elements of $\mathfrak{J}$.

6. We deduce now some of the consequences of the above definition of equality and inequality. In the first place, suppose α, β both lie in $\mathfrak{J}$; *i.e.* let

$$\alpha = (aa', a') = a, \quad \beta = (bb', b') = b,$$

by 4. Now, according to the definition in 5,

$$\alpha \gtreqless \beta$$

according as

$$aa'b' \gtreqless a'bb';$$

that is, according as

$$a \gtreqless b.$$

Thus, when α considered as a number of $\mathfrak{J}$, equals β considered as a number of $\mathfrak{J}$, the two are equal, considered as numbers in $\mathfrak{F}$, and conversely.

7. *If* $\alpha = \gamma$, $\beta = \gamma$, *then* $\alpha = \beta$.

For, let $\alpha = (a, a')$, $\beta = (b, b')$, $\gamma = (c, c')$.

Since $\alpha = \gamma$, $ac' = a'c$, by 5. (1

Since $\beta = \gamma$, $bc' = b'c$. (2

Multiply 1) by b', and 2) by a' and subtract.

Then $$ab'c' = a'bc'.$$

$$\therefore ab' = a'b. \quad \therefore \alpha = \beta,$$

by 5.

8. *The two numbers* (ma, ma') *and* (a, a') *are equal.*

This follows at once from the definition in 5. From this fact we conclude: *We can multiply the first and second constituent of a number without changing its value.*

Conversely:

If the first and second constituents of a number have a common factor, it can be removed without changing the value of the number.

9. *Let* a, a' *be relative prime. For* $\alpha = (a, a')$ *and* $\beta = (b, b')$ *to be equal, it is necessary and sufficient that*

$$b = ta, \ b' = ta'. \qquad (1$$

Obviously if 1) holds, $\alpha = \beta$. The condition 1) is thus *sufficient. It is necessary.* For, from $\alpha = \beta$, we have

$$ab' = a'b. \qquad (2$$

We apply now the property mentioned in 2. Since $a'b$ is divisible by a, by virtue of 2); and since a, a' are relative prime, b must be divisible by a. Say

$$b = ta. \qquad (3$$

Similarly, since ab' is divisible by a', and a, a' are relative prime, b' must be divisible by a'. Say

$$b' = sa'. \qquad (4$$

Putting 3), 4) in 2), we get $s = t$.
Hence 3), 4) give now 1).

10. Our next step is to define the four rational operations on the elements of $\mathfrak{F}$.

We begin by defining the two direct operations.

Let $\alpha = (a, a'), \ \beta = (b, b').$

We define *addition* by the equation,

$$\alpha + \beta = (ab' + a'b, \ a'b'); \qquad (1$$

and *multiplication*, by

$$\alpha\beta = (ab, \ a'b'). \qquad (2$$

It can be shown that the operations just defined enjoy the same properties as those of ordinary fractions. Without stopping to show this in detail, we demonstrate a few of these properties, by way of illustration.

11. Let α, β lie in $\mathfrak{J}$; and say

$$\alpha = (aa', a') = a, \quad \beta = (bb', b') = b.$$

Then $\alpha + \beta$, as defined in 10, 1), should give $a + b$; and $\alpha\beta$, as defined in 10, 2), should give ab.

This is indeed so. For

$$\begin{aligned} \alpha + \beta &= (aa'b' + a'bb', a'b'), \text{ by } \mathbf{10, 1)} \\ &= (a + b, 1), \text{ by } 8 \\ &= a + b, \text{ by } 4. \end{aligned}$$

Similarly,

$$\begin{aligned} \alpha \cdot \beta &= (aa'bb', a'b'), \text{ by } \mathbf{10, 2)} \\ &= (ab, 1), \text{ by } 8 \\ &= ab, \text{ by } 4. \end{aligned}$$

12. *From* $\alpha\gamma = \beta\gamma$, *follows* $\alpha = \beta$.

For, let $\gamma = (c, c')$; we have:

$$\alpha\gamma = (a, a')(c, c') = (ac, a'c'), \text{ by } \mathbf{10, 2)}$$

$$\beta\gamma = (b, b')(c, c') = (bc, b'c').$$

Since by hypothesis $\alpha\gamma = \beta\gamma$, we have

$$acb'c' = a'c'bc, \text{ by } 5.$$

$$\therefore\ ab' = a'b.$$

$$\therefore\ \alpha = \beta, \text{ by } 5.$$

13. We establish now the following relations:

1) $\alpha + \beta > \alpha$.

2) If $\beta > \gamma$, then $\alpha + \beta > \alpha + \gamma$.

3) If $\alpha + \beta = \alpha + \gamma$, then $\beta = \gamma$.

4) If $\alpha > \beta$ and $\beta > \gamma$, then $\alpha > \gamma$.

To prove 1):

$$\alpha+\beta=(a,\ a')+(b,\ b')=(ab'+a'b,\ a'b'),\ \text{by 10, 1).}$$

But
$$a'(ab'+a'b)>aa'b'.$$
$$\therefore\ \alpha+\beta>\alpha,\ \text{by 5.}$$

To prove 2):

$$\alpha+\beta=(ab'+a'b,\ a'b')$$
$$=(ab'c'+a'bc',\ a'b'c'),\ \text{by 8.} \qquad (4$$

Similarly,
$$\alpha+\gamma=(ac'+a'c,\ a'c')$$
$$=(ab'c'+a'b'c,\ a'b'c'). \qquad (5$$

By hypothesis $\beta>\gamma$; hence, by 5,
$$bc'>b'c. \qquad (6$$

Comparing 4), 5), we see the second constituents are equal, while the first constituent in 4) is greater than the first constituent in 5), by virtue of 6). From this follows, by 5, that $\alpha+\beta>\alpha+\gamma$, which is 2).

To prove 3):

Suppose the contrary; then since $\beta\neq\gamma$, either $\beta>\gamma$ or $\beta<\gamma$.
If $\beta>\gamma$, then $\alpha+\beta>\alpha+\gamma$, by 2). (7
If $\beta<\gamma$, then $\alpha+\gamma>\alpha+\beta$, by 2). (8
But both 7), 8) contradict the hypothesis that $\alpha+\beta=\alpha+\gamma$.

To prove 4):

Since $\alpha>\beta$, $$ab'>a'b. \qquad (9$$

Since $\beta>\gamma$, $$bc'>b'c. \qquad (10$$

From 9), 10) we have
$$abb'c'>a'bb'c;$$
whence
$$ac'>a'c.$$

Hence, by 5,
$$\alpha>\gamma.$$

14. As an illustration of the demonstration for the formal laws governing addition and multiplication, let us show that the *distributive law* holds in $\mathfrak{F}$. We wish to prove that

1) $$\alpha(\beta+\gamma)=\alpha\beta+\alpha\gamma.$$

Now $$\beta+\gamma=(bc'+b'c,\ b'c'), \text{ by } 10, 1).$$

$$\alpha(\beta+\gamma)=(a,\ a')\cdot(bc'+b'c,\ b'c')$$

2) $$=(abc'+ab'c,\ a'b'c'), \text{ by } 10, 2).$$

Also

$$\alpha\beta=(ab,\ a'b');\ \alpha\gamma=(ac,\ a'c').$$

$$\therefore\ \alpha\beta+\alpha\gamma=(aa'bc'+aa'b'c,\ a'^2b'c')$$

3) $$=(abc'+ab'c,\ a'b'c'), \text{ by } 8.$$

The comparison of 2), 3) gives 1).

15. We turn now to the inverse operations, subtraction and division; considering first *division*.

We define the quotient of α by β to be the element or elements, ξ, if any exist, of $\mathfrak{F}$ which satisfy the relation

$$\alpha=\beta\xi. \tag{1}$$

Set $\xi=(x,\ x')$. Since ξ must satisfy 1), we have

$$(a,\ a')=(b,\ b')(x,\ x')=(bx,\ b'x').$$

The first and third members give, by 5,

$$ab'x'=a'bx, \tag{2}$$

which x, x' must satisfy.

A solution of 2) is obviously

$$x=ab', \qquad x'=a'b.$$

Thus,

$$\xi=(ab',\ a'b)$$

is a solution of 1). This is the only solution of 1). For, suppose η is a solution. Then by definition

$$\alpha=\beta\eta. \tag{3}$$

Then 1), 3) give, by 7,

$$\beta\xi=\beta\eta,$$

which gives, by 12,

$$\xi=\eta.$$

16. The quotient of α by β, we shall now represent by α/β. All the numbers of $\mathfrak{F}$ may be regarded as quotients of numbers in $\mathfrak{J}$. For, let $\alpha=(a,\,a')$ be any number of $\mathfrak{F}$. It evidently satisfies the equation

$$a = a'x,$$

which, as we have just seen, admits only one root, viz. the quotient of a by a'.

Hence
$$\alpha=(a,\,a')=a/a'.$$

Thus the elements or numbers in $\mathfrak{F}$ are ordinary positive fractions.

17. We have now this result. In the system $\mathfrak{F}$, division is always possible and unique. In the old system $\mathfrak{J}$, this is not true; the division of a by b being only possible when a is a multiple of b. We see, then, that, on properly enlarging our number system by introducing new elements, we obtain a system $\mathfrak{F}$ which has this advantage over $\mathfrak{J}$, that the quotient of any two numbers in $\mathfrak{F}$ exists and is unique.

18. We treat now *subtraction.*

We define the result of subtracting β from α to be the element or elements, call them ξ, in $\mathfrak{F}$, which satisfy the relation

$$\alpha = \beta + \xi. \qquad (1$$

If $\beta \geqq \alpha$, there exists no number ξ in $\mathfrak{F}$ which satisfies 1). For, if $\beta=\alpha$,

$$\beta+\xi=\alpha+\xi>\alpha, \text{ by 13, 1).}$$

If $\beta>\alpha$,
$$\beta+\xi>\alpha+\xi, \text{ by 13, 2).}$$

Also
$$\alpha+\xi>\alpha, \text{ by 13, 1).}$$

$$\therefore\ \beta+\xi>\alpha, \text{ by 13, 4).}$$

Thus when $\beta\geqq\alpha$, $\beta+\xi>\alpha$, and hence $\beta+\xi\neq\alpha$.

Suppose then, that $\beta<\alpha$. Then

$$ab'>a'b. \qquad (2$$

From 1), we have, setting $\xi=(x,\,x')$;

$$\begin{aligned}(a,\,a')&=(b,\,b')+(x,\,x')\\ &=(bx'+b'x,\,b'x'), \text{ by 10, 1).}\end{aligned}$$

Hence by 5, observing 2),

$$a'b'x = x'(ab' - a'b). \tag{3}$$

A solution of 3) is evidently

$$x = ab' - a'b, \ x' = a'b'.$$

Hence

$$\xi = (ab' - a'b, a'b')$$

is a solution of 1).

This is the only solution; for if η is a solution, we have, by definition,

$$\alpha = \beta + \eta. \tag{4}$$

The comparison of 1), 4) gives

$$\beta + \xi = \beta + \eta.$$

Hence, by 13, 3),

$$\xi = \eta.$$

19. We have thus this result: In the system $\mathfrak{F}$, the subtraction of β from α is possible and *unique*, when $\alpha > \beta$; when $\alpha \leq \beta$, it is impossible. That is, there is no number ξ in $\mathfrak{F}$ which satisfies 18, 1). When subtraction is possible, we represent the result of subtracting β from α by $\alpha - \beta$.

Negative Numbers

20. In the system of positive fractions $\mathfrak{F}$, subtraction is only possible when the minuend is greater than the subtrahend. To remove this restriction, we propose to form a new number system R, which contains all the numbers of $\mathfrak{F}$; and in which subtraction of a greater from a less shall be possible. Since the method of forming R is identical with that employed for $\mathfrak{F}$, we shall be more brief now. The numbers in $\mathfrak{F}$ we now denote by $a, b, c, \dots$, while the Greek letters $\alpha, \beta, \gamma, \dots$ shall denote numbers in the new system R.

21. 1. We begin by taking the elements of $\mathfrak{F}$ in pairs, to form new objects, which we denote by the new symbol $\{a, b\}$. The totality of all such pairs forms the system R.

Next, we place R in relation to $\mathfrak{F}$. Let $\alpha = \{a, b\}$. In case $a > b$, we say α shall be the number $a - b$, which obviously lies in

$\mathfrak{F}$. Thus every number in $\mathfrak{F}$ lies in R. For, let a be any number in $\mathfrak{F}$, and let b be any other number in $\mathfrak{F}$. Then

$$\{a + b, b\} = (a + b) - b = a;$$

that is, a lies in R.

2. We next *order* the system R. We say

$$\{a, a'\} \gtreqless \{b, b'\},$$

according as

$$a + b' \gtreqless a' + b. \tag{1}$$

3. *Addition* is defined by the relation

$$\alpha + \beta = \{a, a'\} + \{b, b'\} = \{a + b, a' + b'\}. \tag{2}$$

Multiplication is defined by

$$\alpha \cdot \beta = \{ab + a'b', ab' + a'b\}. \tag{3}$$

4. As a consequence of 1), we have

$$\{a, a'\} = \{a + b, a' + b\}, \tag{4}$$

where b is any number in $\mathfrak{F}$.

In words, 4) states:

We can add the same number b to both constituents of $\alpha = \{a, a'\}$ without changing the value of α; and if a, a' are both $> b$, we can subtract b from both constituents, without altering the value of α.

5. It is easy now to prove results analogous to those in 6, 7, 11, 14; in particular the associative, commutative, and distributive laws.

22. According to our definition of equality, all the elements of R whose first and second constituents are the same, *i.e.* all elements of the type

$$\{a, a\},$$

are equal. We set

$$\{a, a\} = 0,$$

and call this number *zero*.

Then, if in $\alpha = \{a, a'\}$,

$$a > a', \quad \alpha > 0;$$

if $a' > a$,

$$\alpha < 0.$$

Numbers in R which are >0, are called *positive;* those <0, are called *negative.* The number 0 is neither positive nor negative. From this, it follows that the positive numbers of R are simply the numbers of $\mathfrak{F}$; while 0 and all negative numbers do not lie in $\mathfrak{F}$.

23. **1.** We observe that

$$\alpha + 0 = \alpha = 0 + \alpha, \tag{1}$$

$$\alpha \cdot 0 = 0 = 0 \cdot \alpha. \tag{2}$$

To prove 1).

Let $\alpha = \{a, a'\}, \ 0 = \{b, b\}.$

Then $\alpha + 0 = \{a+b, a'+b\}$, by 21, 3;

$= \{a, a'\}$, by 21, 4;

$= \alpha.$

To prove 2).

$$\alpha \cdot 0 = \{ab + a'b, ab + a'b\}$$

$= 0$, by 22.

2. We also note the relations

$$0 + 0 = 0, \quad 0 \cdot 0 = 0.$$

24. We can prove now easily the *Rule of signs. The product of two positive or two negative numbers is positive. The product of a positive and a negative number is negative.*

Let $\alpha = \{a, a'\}, \ \beta = \{b, b'\}.$

1°. $\alpha, \beta > 0$, *then* $\alpha\beta > 0.$

For here $a > a', \ b > b'$, by 22.

$\therefore \ \alpha\beta = \{ab + a'b', ab' + a'b\} = \{b(a-a') + a'b', ab'\}$, by 21, 4

$= \{b(a-a'), b'(a-a')\}$, by 21, 4

>0, since $b(a-a') > b'(a-a')$.

2°. $\alpha, \beta < 0$, *then* $\alpha\beta > 0.$

Here $a' > a, \ b' > b$, by 22.

$$\therefore\ \alpha\beta = \{ab + a'(b'-b),\ ab'\},\ \text{by 21, 4;}$$

$$= \{a'(b'-b),\ a(b'-b)\},\ \text{by 21, 4;}$$

$$> 0,\ \text{since}\ a'(b'-b) > a(b'-b).$$

3°. $\alpha > 0,\ \beta < 0,$ *then* $\alpha\beta < 0.$

Here $a > a',\ b' > b.$

$$\therefore\ \alpha\beta = \{ab + a'(b'-b),\ ab'\}$$

$$= \{a'(b'-b),\ a(b'-b)\}$$

$$< 0,\ \text{since}\ a'(b'-b) < a(b'-b).$$

25. *The product of any two numbers in R vanishes when, and only when, one of the factors is zero.*

Let α, β be any two numbers in R.

We saw in 23 that

$$\alpha\beta = 0,$$

when either α or $\beta = 0$.

Conversely, if $\alpha\beta = 0$, either α or $\beta = 0$.

For, if neither α nor $\beta = 0$, these numbers are either positive or negative. Their product is therefore either positive or negative by 24, and hence not zero. This is a contradiction.

26. Let us consider the following important formulæ, viz.:

1) If $\beta > \gamma$, then $\alpha + \beta > \alpha + \gamma$.

2) From $\alpha + \beta = \alpha + \gamma$, follows $\beta = \gamma$.

3) If $\alpha \neq 0$ and $\alpha\beta = \alpha\gamma$, then $\beta = \gamma$.

To prove 1):

$$\alpha + \beta = \{a + b,\ a' + b'\},\ \text{by 21, 3;}$$

$$= \{a + b + c',\ a' + b' + c'\},\ \text{by 21, 4.} \quad (4$$

Similarly,

$$\alpha + \gamma = \{a + c,\ a' + c'\} = \{a + b' + c,\ a' + b' + c'\}. \quad (5$$

Since $\beta > \gamma$, $b + c' > b' + c$, by 21, 2.

$$\therefore\ a + b + c' > a + b' + c. \quad (6$$

If we now apply the definition for *greater than* given in 21, 2 to $\alpha+\beta$ and $\alpha+\gamma$, the relations 4), 5), 6) show that $\alpha+\beta>\alpha+\gamma$.

To prove 2):

Suppose the contrary, *i.e.* suppose $\beta>\gamma$ or $\beta<\gamma$.

If $\beta>\gamma$, $\alpha+\beta>\alpha+\gamma$, by 1).

If $\gamma>\beta$, $\alpha+\gamma>\alpha+\beta$, by 1).

Thus in both cases, $\alpha+\beta\neq\alpha+\gamma$, which is contrary to hypothesis. Hence $\beta=\gamma$.

To prove 3):

From $\alpha\beta=\alpha\gamma$ we have

$$\alpha(\beta-\gamma)=0.$$

Applying 25, we have $\beta=\gamma$.

27. We turn now to *subtraction*. This we define as in $\mathfrak{F}$, viz.: the result of subtracting β from α is the element or elements ξ, of R, which satisfy

$$\alpha=\beta+\xi. \tag{1}$$

This equation gives, setting $\xi=\{x, x'\}$,

$$\{a, a'\}=\{b, b'\}+\{x, x'\}=\{b+x, b'+x'\}, \text{ by 21, 3.}$$

Hence by 21, 2,

$$a+b'+x'=a'+b+x.$$

This equation is evidently satisfied by

$$x=a+b',\ x'=a'+b.$$

Hence

$$\xi=\{a+b',\ a'+b\} \tag{2}$$

is a solution of 1).

This is the only solution. For, let η be a solution. Then by definition,

$$\alpha=\beta+\eta. \tag{3}$$

The comparison of 1), 3) gives

$$\beta+\xi=\beta+\eta.$$

Hence by 26, 2),

$$\xi=\eta.$$

28. We have thus this result: in the system R subtraction is always *possible* and *unique*. The result of subtracting β from α, we represent by $\alpha - \beta$; it is a number in R. Then any number $\alpha = \{a, a'\}$ in R, is the result of subtracting a' from a, or

$$\alpha = a - a'.$$

For,

$$a = \{a + b, b\}, \text{ by 21, 4.}$$

Similarly,

$$a' = \{a' + b, b\}.$$

Hence

$$\begin{aligned} a - a' &= \{a + b, b\} - \{a' + b, b\} \\ &= \{a + 2b, a' + 2b\}, \text{ by 27, 2)} \\ &= \{a, a'\}, \text{ by 21, 4.} \end{aligned}$$

29. 1. Let $\alpha = \{a, a'\}$ be any number of R.
The number $\{a', a\}$ is called *minus* α, and we write

$$\{a', a\} = -\alpha.$$

Then

$$-(-\alpha) = -\{a', a\} = \{a, a'\} = \alpha.$$

Also,

$$\begin{aligned} \alpha + (-\alpha) &= \{a, a'\} + \{a', a\} \\ &= \{a + a', a + a'\} = 0 \\ &= \alpha - \alpha. \end{aligned}$$

If α is positive, $-\alpha$ is negative; and conversely, if α is negative, $-\alpha$ is positive.

2. The number $-\alpha$ may be defined as the number ξ, such that

$$\alpha + \xi = 0.$$

For, $\xi = -\alpha$ satisfies this equation; and, as we saw in 27, this equation admits but one solution. This shows that the numbers in R, $\neq 0$, may be grouped in pairs, such that their sum is zero.

3. If $-\alpha = -\beta$, then $\alpha = \beta$.
For, multiplying both sides of $-\alpha = -\beta$ by -1, we get $\alpha = \beta$.

4. Every number α, of R, different from zero, can be written in the form

$$\alpha = a, \text{ or } \alpha = -a,$$

where a is a number in $\mathfrak{F}$.

For if $\alpha > 0$, we already know by 22 that α is a number in $\mathfrak{F}$. If $\alpha < 0$, then $-\alpha$ is positive, so that $-\alpha = a$, a number in $\mathfrak{F}$ Multiplying this equation by -1, we get

$$\alpha = -a.$$

30. 1. We treat now *division*.

We say: the result of dividing α by β is the number or numbers ξ, of R, such that

$$\alpha = \xi\beta. \tag{1}$$

Suppose $\beta \neq 0$; *then there is one and only one number* ξ; i.e. *in this case, division is possible and unique.*

There can be at most one. For, if η satisfies 1), we should have

$$\alpha = \eta\beta. \tag{2}$$

Comparing 1), 2), we have

$$\xi\beta = \eta\beta;$$

whence by 26, 3),

$$\xi = \eta.$$

To show that there is always *one* solution of 1), we have the following cases.

Let $\alpha, \beta > 0$; then $\alpha = a$, $\beta = b$, by 29, 4; and 1) becomes

$$a = \xi b. \tag{3}$$

But by 15 the solution of 3) is

$$\xi = (a, b) = a/b.$$

Let $\alpha, \beta < 0$; then

$$\alpha = -a, \quad \beta = -b, \text{ by 29, 4.}$$

Then 1) becomes

$$-a = -b\xi;$$

or by 29, 3,

$$a = b\xi.$$

Hence as before,

$$\xi = a/b.$$

Let $\alpha > 0$, $\beta < 0$; then $\alpha = a$. $\beta = -b$, and 1) becomes

$$a = -b\xi. \tag{4}$$

Set $-\xi = \eta$; then 4) gives

$$a = b\eta.$$

Hence,

$$\eta = a/b$$

and

$$\xi = -a/b.$$

If $a < 0$, *and* $\beta > 0$, we get again

$$\xi = -a/b.$$

Finally, let $\alpha = 0$; then 1) becomes

$$0 = \beta\xi.$$

Hence by 25,

$$\xi = 0.$$

2. *We consider now the case that* $\beta = 0$.

The equation 1) admits now no solution, unless $\alpha = 0$ also. For, when $\beta = 0$, $\beta\xi = 0$, whatever ξ may be.

If now $\alpha = 0$, the equation 1) is satisfied for every number ξ in R. We have thus this result: *When the divisor is zero, division is either impossible or entirely indeterminate.*

For this reason, division by zero is excluded in modern mathematics. The admission of division by zero by the older mathematicians, Euler for example, has caused untold confusion. We shall see it is entirely superfluous.

Some Properties of the System R

31. The system R, which we have just formed, is made up of the totality of positive and negative integers and fractions, and also zero. It is called the *system of rational numbers;* any element in it being called a *rational number.* The elementary arithmetical properties of these numbers having been established, there is no further occasion to employ the special notations (a, b) and $\{a, b\}$; we shall, instead, employ the customary ones. Furthermore, we shall represent for the rest of this chapter the numbers in R indifferently by Greek and Latin letters $a, b, c, \cdots \alpha, \beta, \gamma, \cdots$

For the sake of completeness we now proceed to deduce a few properties of R, although the reader is probably familiar with them.

32. *The system R is invariant with respect to the four rational operations.*

This simply means that the addition, subtraction, multiplication, and division of any two elements of R, division by 0 of course excluded, always leads to an element in R.

We saw this is not true for the systems $\mathfrak{J}$ and $\mathfrak{F}$.

33. 1. *The system R is dense.*

This term, taken from the theory of aggregates, which we shall take up later, simply means that between any two numbers a, b in R, exists a third and hence an infinity of numbers.

For, let

$$a = \frac{a_1}{a_2},\ b = \frac{b_1}{b_2};$$

and say $a > b$.

Then

$$d = a_1 b_2 - a_2 b_1$$

is an integer ≥ 1.

Let e be a positive integer. By taking it large enough, we can make

$$ed > n,$$

where n is an arbitrarily large positive integer.

Let h be any integer, such that

$$ea_2 b_1 < h < ea_1 b_2.$$

Then

$$\frac{h}{eb_1 b_2}$$

lies between a, b and represents at least n numbers.

2. *The system $\mathfrak{J}$ is not dense.* For, if we take $a = n$ and $b = n + 1$, no element of $\mathfrak{J}$ lies between a and b.

From this results a remarkable difference between $\mathfrak{J}$ and R. After any element n of $\mathfrak{J}$ follows a certain *next* element, viz. $n + 1$. Not so in R. If a is any number of R, there exists no *next* number to a. For, if b were that number, there would lie no number of R between a and b. But since R is dense, there lie an infinity of numbers of R between a, b.

34. 1. *The system R is an Archimedian system.* That is, there is no positive number a in R so small but that some multiple of a, say na, is greater than any prescribed positive number b of R.

For, let

$$a=\frac{a_1}{a_2}.$$

Let us choose n so large that

$$n>a_2 b.$$

Then

$$na=\frac{na_1}{a_2}>\frac{a_1 a_2 b}{a_2}>a_1 b\geq b.$$

2. *Let a be an arbitrarily large number of R; there exists a positive integer n, such that* $a/n<b$, *where b is arbitrarily small.*

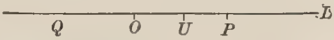
For, by 1, there exists a positive n, such that

$$nb>a.$$

Hence

$$a/n<b.$$

35. Let us lay off the numbers of R on a right line L, just as is done in analytic geometry.



Thus, having chosen an arbitrary point O as origin, and an arbitrary segment OU as the unit of length, to the positive number p in R, corresponds the point P on L to the right of O, and at a distance p from O. To the negative number $-p$, corresponds the point Q, lying to the left of O, and such that $\overline{OQ}=p$. To zero corresponds the origin O.

Let $a, b, c, \cdots$ be numbers of R, to which correspond the points $A, B, C, \cdots$ on L. If $a<b$, the point B lies to the right of A; if $a<b<c$, the point B lies between A and C.

The point A, corresponding to the number a, is called the *representation* or *image* of a.

The points corresponding to the numbers in R we call *rational points*.

Then, since R is dense, the aggregate $\mathfrak{A}$ formed of the totality of rational points is also *dense*. That is, if P, Q be any two points of $\mathfrak{A}$, there lie an infinity of points between P, Q which belong to $\mathfrak{A}$.

Furthermore, if P be any point of $\mathfrak{A}$, there is no *next* point to P.

This representation of the numbers of R by points on a right line is of great assistance to us in our reasoning, as we shall see.

Some Inequalities

36. We have seen in 29, 4 that every number $a \neq 0$ in R, may be written

$$a = \pm a_0,$$

where a_0 is a positive number.

The *numerical or absolute value of* a is a_0, and is denoted by

$$|a|.$$

We have thus,

$$|a| = a_0.$$

We also set

$$|0| = 0.$$

For example :

$$|-\tfrac{4}{5}| = \tfrac{4}{5}; \qquad |+\tfrac{4}{5}| = \tfrac{4}{5};$$

$$|3-7| = |7-3| = 4.$$

37. 1. We have now the following fundamental relations :

$$|a| = |-a|; \qquad (1$$

$$|a-b| = |b-a|; \qquad (2$$

$$|a \pm b| \leq |a| + |b|; \qquad (3$$

$$|a \pm b| \overset{=}{>} ||a| - |b||; \qquad (4$$

$$|ab| = |a| \cdot |b|; \qquad (5$$

$$\left|\frac{a}{b}\right| = \frac{|a|}{|b|}, \qquad b \neq 0. \qquad (6$$

They are readily proved. For example, consider 3).

There are various cases, according as a, b are positive, negative, or zero. We treat one specimen case.

Let $a > 0,\ b < 0.$ Let $b = -b_0,\ b_0 > 0.$

Then $a + b = a - b_0, \quad a - b = a + b_0.$

If $a > b_0$, $|a + b| = |a - b_0| = a - b_0 < a + b_0 = |a| + |b|.$

If $a < b_0$, $|a + b| = |a - b_0| = b_0 - a < a + b_0 = |a| + |b|.$

$$|a - b| = |a + b_0| = a + b_0 = |a| + |b|.$$

Which establishes 3) for this case. The other cases are treated similarly.

2. By repeated applications of 3), 5) we get

$$|a_1 \pm a_2 \pm \cdots \pm a_m| \leq |a_1| + \cdots + |a_m|; \tag*{(7}$$

$$|a_1 \cdot a_2 \cdots a_m| = |a_1|\,|a_2| \cdots |a_m|. \tag*{(8}$$

3. Let $A > 0$, and $|a| < A.$ (9

Then from 9) follows

$$-A < a < A; \tag*{(10}$$

and conversely from 10) follows 9).

38. 1. An important relation is the following:

Let $|a - b| < A, \quad |b - c| < B.$ (1

Then $|a - c| < A + B.$ (2

For, from 1) we have, by 37, 3,

$$-A < a - b < A;$$

$$-B < b - c < B.$$

Adding,

$$-(A + B) < a - c < A + B,$$

which gives 2).

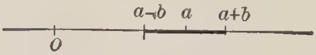
2. A special but common case of the above is when $A = B$; then

$$|a - c| < 2A. \tag*{(3}$$

We shall say that 2) or 3) is obtained from 1) *by adding*.

39. It will be useful to bear in mind the geometric interpretation or image of certain inequalities which recur constantly in the following.

Let a be an arbitrary rational number.



On either side of the point a let us mark off points $a-b$, $a+b$, at a distance b from a. The rational numbers x whose images fall in the interval $a-b$, $a+b$, evidently satisfy the relation

$$a-b \leq x \leq a+b;$$

or what is the same,

$$-b \leq x-a \leq b.$$

That is, x satisfies the inequality

$$|x-a| \leq b. \tag{1}$$

Conversely, the images of the rational numbers x which satisfy 1) lie in the interval $a-b$, $a+b$.

Similarly, the images of the rational numbers x which satisfy

$$0 \leq x-a \leq b,$$

lie in the interval a, $a+b$; while those corresponding to

$$0 \leq a-x \leq b$$

lie in $a-b$, a.

Rational Limits

40. In the next chapter we shall have a good deal to say of infinite sequences and their limits. We proceed to define them as far as rational numbers are concerned.

Let A be a set of rational numbers such that:

1°. It is determined whether a given number belongs to A or not.

2°. There is a first number a_1, of the set, a second number a_2; and in general, after each a_n follows a certain number a_{n+1}.

The set A is then called an *infinite sequence* or simply a *sequence*, and is denoted by

$$A = a_1,\ a_2,\ \cdots \text{ or by } A = \{a_n\}.$$

EXAMPLES

41. 1. If we take $a_n = n$,

$$A = 1, 2, 3, \cdots = \{n\}$$

is a sequence.

2. If we take $a_n = \frac{1}{n}$, we get a sequence

$$A = 1, \frac{1}{2}, \frac{1}{3}, \cdots = \left\{\frac{1}{n}\right\}.$$

3. If $a_n = 1$, we get a sequence

$$A = 1, 1, 1, 1, \cdots = \{1\}.$$

4. Let A consist of the rational numbers lying in the interval 0, 1, arranged in order of magnitude.

This set A is not a sequence. For, although it is perfectly determined what numbers belong to A, and although there is a first element $a_1 = 0$, there is no second element, no third element, etc., by 33, 2. Thus, while condition 1° is satisfied, condition 2° is not.

42. 1. We define now the term *limit*.

Let l be a fixed rational number. We say: l is the limit of the sequence $A = \{a_n\}$, when for each positive rational number ϵ, small at pleasure, there exists an index m, such that

$$|l - a_n| < \epsilon \tag{1}$$

for every $n > m$.

In symbols we write

$$l = \lim_{n=\infty} a_n;$$

we also use the shorter forms

$$l = \lim a_n, \quad \text{or} \quad a_n \doteq l,$$

when no confusion can arise.

We shall also employ at times the symbol

$$l = \lim_{A} a_n.$$

When l is the limit of A, we say A is a *convergent* sequence, and that a_n converges to l as a limit.

2. *Notation.* We shall find it extremely convenient to employ the following abbreviation:

$$\epsilon > 0, \quad m, \quad |l - a_n| < \epsilon, \quad n > m \tag{2}$$

to mean that, *for each positive rational ϵ there exists an index m, such that $|l - a_n| < \epsilon$ for every $n > m$.*

The reader should therefore repeat the italics often enough to himself to be able to read the line of symbols 2) without hesitation.

3. The reader should observe that from 2) we can conclude also that for each positive M there exists an m', such that

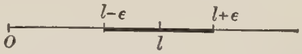
$$|l-a_n|<\frac{\epsilon}{M}, \qquad n>m'.$$

If, therefore, $\{a_n\}$ has the rational limit l, we can write

$$\epsilon>0, \quad m, \quad |l-a_n|<\frac{\epsilon}{M}, \quad n>m. \tag*{(3}$$

We have, of course, changed the notation slightly in 3) by dropping the accent of m'.

43. The graphical interpretation of this definition will prove most helpful in our subsequent reasoning.

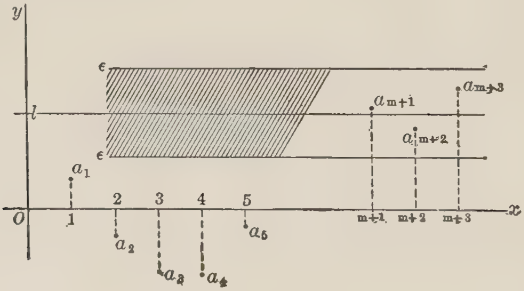


Let us lay off the points on our axis, corresponding to the numbers a_n, also the point corresponding to l. On either side of l lay off the points $l-\epsilon$, $l+\epsilon$. These determine an interval, marked heavy in the figure, which we shall call the *ε-interval.*

If now l is the limit of the sequence A, there must exist for each little ε-interval, an index m, such that the images of all the numbers a_{m+1}, a_{m+2}, $\cdots$ fall within the ε-interval. See 39.

In general, as ϵ is taken smaller and smaller, the index m increases. The definition, however, only requires that for each given ϵ there exists some corresponding m such that 42, 1) holds for every n greater than this m.

44. Another useful graphical interpretation of the definition of a limit is the following.



We take two axes x, y as in analytic geometry. On the x-axis mark off points 1, 2, 3, ⋯ at equal distances apart. Lay off the numbers a_1, a_2, a_3, ⋯ as ordinates on lines through the points 1, 2, 3, ⋯ parallel to the y-axis. (See Fig.) These points we may consider as the images of the numbers a_n. On either side of the line $y = l$, draw parallel lines at a distance ϵ from it. We get then a band, shaded in the figure, which we shall call the *ϵ-band.*

Then, if l is the limit of A, there exists for each ϵ an index m, such that the images of all the numbers a_{m+1}, a_{m+2}, ⋯ fall within the corresponding ϵ-band.

45. EXAMPLES

1. $$A = \left\{\frac{1}{n}\right\}; \ \lim a_n = \lim \frac{1}{n} = 0.$$

2. $$A = \left\{1 - \frac{1}{n}\right\}; \ \lim \left(1 - \frac{1}{n}\right) = 1.$$

3. $$A = 1, -\tfrac{1}{2}, +\tfrac{1}{3}, -\tfrac{1}{4}, \cdots \ a_n = (-1)^{n+1}\frac{1}{n}; \ \lim a_n = 0.$$

The reader will find it helpful to construct the graphs, explained in 43, 44, for each of these sequences.

46. *If it is known of two rational numbers p, q, that $|p - q| < \epsilon$, however small $\epsilon > 0$ may be taken, then $p = q$.*

For, if $p \neq q$, say $p > q$, then $p - q$ is a definite positive rational number; call it d. Then $|p - q|$ is not $< d$, and this contradicts the hypothesis. Hence $p = q$.

47. *A rational sequence $A = \{a_n\}$ cannot have two rational limits l, l'.*

For, since $a_n \doteq l$, we have by definition,

$$\epsilon > 0, \quad m_1, \quad |l - a_n| < \epsilon, \quad n > m_1. \tag{1}$$

Also, since $a_n \doteq l'$, we have

$$\epsilon > 0, \quad m_2, \quad |l' - a_n| < \epsilon, \quad n > m_2. \tag{2}$$

Let $m > m_1, m_2$; then from 1), 2) follows

$$\epsilon > 0, \quad m, \quad |l - a_n| < \epsilon, \quad n > m. \tag{3}$$

$$\epsilon > 0, \quad m, \quad |l' - a_n| < \epsilon, \quad n > m. \tag{4}$$

The inequalities 3), 4), holding now for the same m, we can add them, and get, by 38, 3),

$$|l - l'| < 2\epsilon. \qquad (5$$

But since ϵ is small at pleasure, so is 2ϵ. The inequality 5) gives, by 46,

$$l = l'.$$

48. *If the rational sequence* $\{a_n\}$ *has a rational limit* l, *there exists an index* m, *such that*

$$b < a_n < c; \quad n > m, \qquad (1$$

where b *is any rational number* $< l$, *and* c *any rational number* $> l$.

For, since $a_n \doteq l$,

$$\epsilon > 0, \ m, \ |l - a_n| < \epsilon. \quad n > m.$$

$$\therefore \ l - \epsilon < a_n < l + \epsilon. \qquad (2$$

Since ϵ is arbitrarily small, we can take it so small that

$$l - \epsilon > b, \ l + \epsilon < c.$$

Then 2) gives 1).

49. *Let the two sequences* $\{a_n\}$, $\{b_n\}$ *have the rational limits* a, b *respectively. Then*

$$\lim (a_n + b_n) = a + b; \ \lim (a_n - b_n) = a - b.$$

For, $|(a + b) - (a_n + b_n)| = |(a - a_n) + (b - b_n)|$

$$\leq |a - a_n| + |b - b_n| \qquad (1$$

by 37, 3).

Since $a_n \doteq a$, we have

$$\epsilon > 0, \ m', \ |a - a_n| < \epsilon/2. \quad n > m'. \qquad (2$$

Since $b_n \doteq b$, we have

$$\epsilon > 0, \ m'', \ |b - b_n| < \epsilon/2. \quad n > m''. \qquad (3$$

By choosing m so large that $m > m', m''$, we can suppose 2), 3) hold for the same m.*

---

* When $a > b, \ a > c, \ a > d \cdots$
we shall often set more shortly $a > b, \ c, \ d \cdots$
Similarly $a \neq 0, \ b \neq 0, \ c \neq 0 \cdots$
may be written more shortly $a, \ b, \ c, \cdots \neq 0$

Then 1) becomes, using 2), 3),

$$|(a+b)-(a_n+b_n)|<\frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon. \qquad n>m.$$

This states that

$$\lim (a_n+b_n)=a+b.$$

Similarly, we prove the other half of our theorem.

50. *If the two sequences $\{a_n\}$, $\{b_n\}$ have the rational limits a, b respectively, then*

$$\lim a_n b_n = ab. \tag{1}$$

For,

$$d_n = ab - a_n b_n = a(b-b_n)+b_n(a-a_n).$$

$$\therefore\ |d_n| \le |a|\,|b-b_n| + |b_n|\,|a-a_n|, \tag{2}$$

by 37, 3), 5).

Since $b_n \doteq b$, we have, by 48,

$$|b_n| < B. \qquad n>m'.$$

Also, by 42, 3,

$$|b-b_n|<\frac{\epsilon}{2|a|}. \qquad n>m''.$$

Since $a_n \doteq a$, we have, by 42, 3,

$$|a-a_n|<\frac{\epsilon}{2B}. \qquad n>m'''.$$

Evidently by taking m large enough, we can use the same m in these three inequalities.

Then they give in 2)

$$|d_n| < |a|\frac{\epsilon}{2|a|}+B\cdot\frac{\epsilon}{2B}=\frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon,$$

which proves 1).

51. *Let the two sequences $\{a_n\}$, $\{b_n\}$, have the rational limits a, b respectively. Let b and $b_n \neq 0$.*

Then

$$\lim \frac{a_n}{b_n}=\frac{a}{b}. \tag{1}$$

For,

$$d_n = \frac{a}{b} - \frac{a_n}{b_n} = \frac{ab_n - ba_n}{bb_n} = \frac{(ab_n - ab) + (ab - a_n b)}{bb_n}$$

$$= \frac{a(b_n - b)}{bb_n} + \frac{a - a_n}{b_n}.$$

$$\therefore\ |d_n| \leq \frac{|a|}{|b||b_n|}|b - b_n| + \frac{|a - a_n|}{|b_n|}. \qquad (2$$

Since $b \neq 0$, $|b| > 0$. Let B be a rational number, such that

$$0 < B < |b|.$$

Then, by 48, there exists an m, such that

$$|b_n| > B. \qquad n > m. \qquad (3$$

Also, by 42, 3,

$$\epsilon > 0,\ m,\ |b - b_n| < \frac{\epsilon |b| B}{2|a|}. \qquad n > m. \qquad (4$$

$$\epsilon > 0,\ m,\ |a - a_n| < \frac{\epsilon B}{2}. \qquad n > m. \qquad (5$$

By taking m large enough, we can use the same m in these inequalities. Putting 3) in 2), we get

$$|d| < \frac{|a|}{|b|B}|b - b_n| + \frac{|a - a_n|}{B}$$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \text{ by 4), 5)},$$

which proves 1).

CHAPTER II

IRRATIONAL NUMBERS

Insufficiency of R

52. Although the system of rational numbers R is dense, and so *apparently complete*, it is easy to show that it is quite insufficient for the needs of even elementary mathematics.

Consider, for example, the length δ of the diagonal of a unit square. This length is defined by the equation

$$\delta^2 = 2. \tag{1}$$

We can show there is no number in R which satisfies 1). For, suppose

$$\delta = \frac{a}{b},$$

where a, b are two positive integers, which we can take without loss of generality, relatively prime.

Then 1) gives

$$a^2 = 2\,b^2.$$

Let p be any prime factor of b. It is then a divisor of a^2, and so of a. Thus a and b are both divisible by p. They are thus not relatively prime, unless $p = 1$.

Thus $b = 1$; and δ is an integer. But obviously there is no integer whose square is 2.

53. 1. A similar reasoning shows that

$$\sqrt[n]{a}$$

does not lie in $\boldsymbol{R}$, unless a is the nth power of a rational number.

The numbers

$$e = 2.71828\cdots, \quad \pi = 3.14159\cdots$$

can be shown to be irrational; the numbers

$$\log x, \quad e^x, \quad \sin x, \quad \tan x$$

are in general not rational.

2. Let us show that

$$l = \log 5,$$

the base being 10, does not lie in R.

If l were rational, we should have

$$l = \frac{a}{b},$$

where a, b are integers.

Then

$$10^{\frac{a}{b}} = 5. \quad \therefore \; 10^a = 5^b. \qquad (1$$

Obviously l cannot be negative; we can thus suppose $a, b > 0$.

Now any integral positive power of 10 is an integer ending in 0; while any integral positive power of 5 ends in 5.

Thus 1) requires that a number ending in 0 should equal a number ending in 5, which is absurd. Hence l is not rational.

Cantor's Theory

54. 1. The preceding remarks show clearly the necessity of forming a more comprehensive system of numbers than R. How this may be done in various ways has been shown by Weierstrass, Cantor, Dedekind, Hilbert, and others.

We adduce now certain considerations which lead up to Cantor's theory.

We have seen no rational number exists which satisfies the equation

$$x^2 = 2. \qquad (1$$

It is, however, possible to determine an infinite sequence of rational numbers

$$a_1, \; a_2, \; a_3, \; \cdots \qquad (A$$

such that

$$\lim a_n^2 = 2.$$

The method we now give for finding such a sequence A has no practical value; it has, however, theoretical importance.

For a_1, we take the greatest integer, such that

$$a_1^2 < 2.$$

In the present case, $a_1 = 1$.

From the numbers

$$a_1 + \tfrac{1}{10},\ a_1 + \tfrac{2}{10},\ \cdots\ a_1 + \tfrac{9}{10}, \tag{2}$$

we take for a_2 the number whose square is < 2, while the next number of 2) gives a square > 2.

Suppose

$$a_2 = a_1 + \frac{\alpha_1}{10}.$$

Then

$$a_2^2 < 2 < (a_2 + \tfrac{1}{10})^2.$$

From the numbers

$$a_2 + \frac{1}{10^2},\ a_2 + \frac{2}{10^2},\ \cdots\ a_2 + \frac{9}{10^2}, \tag{3}$$

we take for a_3 the number whose square is < 2, while the next number of 3) gives a square > 2.

Suppose

$$a_3 = a_2 + \frac{\alpha_2}{10^2}.$$

Then

$$a_3^2 < 2 < \left(a_3 + \frac{1}{10^2}\right)^2.$$

We may proceed in this way without end, and get thus an infinite sequence of rational numbers,

$$\begin{aligned}
&a_1;\\
&a_2 = a_1 + \frac{\alpha_1}{10};\\
&a_3 = a_2 + \frac{\alpha_2}{10^2} = a_1 + \frac{\alpha_1}{10} + \frac{\alpha_2}{10^2};\\
&a_4 = a_3 + \frac{\alpha_3}{10^3} = a_1 + \frac{\alpha_1}{10} + \frac{\alpha_2}{10^2} + \frac{\alpha_3}{10^3};\\
&\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\
&a_n = a_{n-1} + \frac{\alpha_{n-1}}{10^{n-1}} = a_1 + \frac{\alpha_1}{10} + \frac{\alpha_2}{10^2} + \cdots + \frac{\alpha_{n-1}}{10^{n-1}}.\\
&\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot
\end{aligned}$$

By actual calculation we find the numbers

$$a_1, \quad a_2, \quad a_3, \quad a_4, \quad \cdots$$

are respectively

$$1, \quad 1.4, \quad 1.41, \quad 1.414, \quad 1.4142, \quad \cdots$$

2. We show now that

$$\lim a_n^2 = 2.$$

For, from

$$a_n^2 < 2 < \left(a_n + \frac{1}{10^{n-1}}\right)^2,$$

we have

$$0 < 2 - a_n^2 < \left(a_n + \frac{1}{10^{n-1}}\right)^2 - a_n^2.$$

$$\therefore \ |2 - a_n^2| < \frac{3}{10^{n-1}} + \frac{1}{10^{2(n-1)}}.$$

Obviously now, for each rational $\epsilon > 0$, we can find an m, such that

$$\frac{3}{10^{m-1}} + \frac{1}{10^{2(m-1)}} < \epsilon.$$

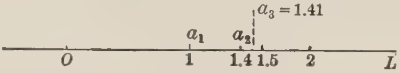
Then

$$|2 - a_n^2| < \epsilon. \qquad n > m.$$

Hence

$$\lim a_n^2 = 2.$$

55. 1. The method given in 54 for forming the sequence a_1, a_2, a_3, $\cdots$ admits a simple graphical interpretation.



We first divide the indefinite right line L into unit segments; a_1 is end point of one of these segments. In the present case $a_1 = 1$.

We next divide the segment 1, 2 into 10 equal parts; a_2 is the end point of one of these segments. In the present case $a_2 = 1.4$.

We next divide the segment 1.4, 1.5 into 10 equal parts; a_3 is the end point of one of these segments. In this way, we continue subdividing each successive little interval or segment into 10 smaller parts, without end.

We observe that each little segment is contained in the immediately preceding one, and therefore in all preceding ones.

Also, that the lengths of these segments form a sequence

$$1,\ \frac{1}{10},\ \frac{1}{10^2},\ \frac{1}{10^3},\ \cdots$$

whose limit is zero.

56. 1. The method of 54 may be used to find an infinite sequence of rational numbers

$$a_1,\quad a_2,\quad a_3,\quad \cdots$$

which more and more nearly satisfy the equation

$$10^x = 5,$$

which defines log 5.

We find:

$$a_1 = 0,\quad a_2 = .6,\quad a_3 = .69,\quad a_4 = .698,\quad \cdots$$

2. The same method may evidently be applied to any problem which defines an *irrational* number. In each case it leads to a sequence of rational numbers

$$a_1,\quad a_2,\quad a_3,\quad \cdots \qquad A$$

such that

1°. Each number a_n satisfies more nearly than the preceding ones the conditions of the problem.

2°. For each positive rational ϵ, arbitrarily small, there exists an index m, such that

$$|a_n - a_\nu| < \epsilon,$$

for every $n,\ \nu \geq m$.

57. *Regular Sequences.* 1. It is this second property of the sequences A, that Cantor seizes on to construct the elements of his number system. We lay down now the following *definitions.*

Any infinite sequence of rational numbers

$$a_1,\ a_2,\ a_3,\ \cdots$$

which has property 2° in 56 is called *regular.*

As in 42, 2, we shall indicate this property by the abbreviated notation:

$$\epsilon > 0,\quad m,\quad |a_n - a_\nu| < \epsilon,\quad n,\ \nu \geq m. \qquad (1$$

2. Every regular sequence *defines a number*, which we represent by the symbol

$$\alpha = (a_1, \ a_2, \ a_3, \ \cdots).$$

The totality of such numbers forms a *number system*, called the *system of real numbers*.

We shall denote it by $\mathfrak{R}$, which may be read *German R.*

For the convenience of the reader, we shall denote in this chapter the new numbers, *i.e.* the numbers in $\mathfrak{R}$, by the Greek letters $\alpha, \beta, \gamma, \cdots$; while the Latin letters $a, b, c, \cdots$ denote numbers in the old system R.

To see if a given sequence is *regular*, we must see if the inequalities 1) are satisfied. For this reason we shall speak of these inequalities as *the ϵ, m test.*

3. The ϵ, m test is equivalent to the following:

$$\epsilon > 0, \quad m, \quad |a_n - a_m| < \epsilon, \quad n > m. \tag{2}$$

The difference between 1), 2) being that in $|a_n - a_m|$, only one index, n, varies.

For, when 1) *holds,* 2) *is satisfied.* For we pass from 1) to 2) by setting $\nu = m$ in 1).

Conversely, if 2) *holds,* 1) *is satisfied.*

For, since ϵ in 2) is small at pleasure, let us take

$$\epsilon < \frac{\sigma}{2}.$$

Then 2) gives

$$|a_n - a_m| < \frac{\sigma}{2}. \qquad n > m.$$

Also

$$|a_\nu - a_m| < \frac{\sigma}{2}. \qquad \nu \geq m.$$

Adding the inequalities, we get, by 38, 3),

$$|a_n - a_\nu| < \sigma, \quad n, \nu \geq m,$$

which is 1).

4. We observe finally that we may replace $n, \nu \geq m$ in 1) by $n, \nu > m$.

For, if

$$|a_n - a_\nu| < \epsilon, \tag{3}$$

for every $n, \nu \geqq m$, it is true for every $n, \nu > m$. Conversely, if 3) is true for every $n, \nu > m$, it is true for every $n, \nu \geqq m + 1$. We would therefore in 1) replace m by $m + 1$.

58. EXAMPLES

1. That the sequences A, defined in 54, are *regular*, is readily shown. We have

$$a_n = a_1 + \frac{\alpha_1}{10} + \cdots + \frac{\alpha_{n-1}}{10^{n-1}}.$$

$$a_\nu = a_1 + \frac{\alpha_1}{10} + \cdots + \frac{\alpha_{\nu-1}}{10^{\nu-1}}.$$

For simplicity, suppose $\nu > n$;

then

$$a_\nu - a_n = \frac{\alpha_n}{10^n} + \frac{\alpha_{n+1}}{10^{n+1}} + \cdots + \frac{\alpha_{\nu-1}}{10^{\nu-1}} < \frac{1}{10^{n-1}}, \tag{1}$$

as the considerations of 55 show.

If we choose m so large that

$$\frac{1}{10^{m-1}} < \epsilon,$$

then, by 1),

$$a_\nu - a_n < \epsilon. \qquad n, \nu \geq m.$$

The ϵ, m test is therefore satisfied.

2. Consider the sequence

$$\frac{1}{1}, \ -\frac{1}{2}, \ \frac{1}{3}, \ -\frac{1}{4}, \ \cdots$$

Here

$$|a_n - a_\nu| = \left|\frac{1}{n} \pm \frac{1}{\nu}\right| \leqq \frac{1}{n} + \frac{1}{\nu}. \tag{2}$$

If we take now

$$m > \frac{2}{\epsilon},$$

then

$$\frac{1}{n} + \frac{1}{\nu} < \frac{1}{m} + \frac{1}{m}. \qquad n, \nu > m.$$

Hence 2) gives

$$|a_n - a_\nu| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

3. Consider the sequence

$$1, \ 1, \ 1, \ 1, \ 1, \ \cdots$$

Here

$$a_n - a_\nu = 1 - 1 = 0,$$

and this sequence evidently satisfies the ϵ, m test, and is therefore regular.

4. Consider $1,\ -1,\ 1,\ -1,\ \cdots$

Here $|a_n - a_\nu| = 0$ or 2.

Evidently no m exists, such that

$$|a_n - a_\nu| < \epsilon. \qquad n, \nu > m.$$

The sequence is thus *not* regular.

5. Consider $1,\ 2,\ 3,\ 4,\ \cdots$

Here $|a_n - a_\nu| = |n - \nu|$,

and the ϵ, m test is obviously not satisfied. The sequence is therefore *not* regular.

59. *For any regular sequence of rational numbers* $a_1, a_2, \cdots$ *there exists a positive number* M, *such that*

$$|a_n| < M. \qquad n = 1, 2, 3, \cdots \tag{1}$$

For, the sequence being regular,

$$\epsilon > 0,\ m,\ |a_n - a_m| < \epsilon. \qquad n > m.$$

Hence

$$a_m - \epsilon < a_n < a_m + \epsilon. \tag{2}$$

Let M be taken greater than any of the $m + 2$ numbers.

$$|a_1|,\ |a_2|\ \cdots\ |a_m|,\ |a_m - \epsilon|,\ |a_m + \epsilon|.$$

Then 2) proves 1).

60. The elements of $\mathfrak{R}$ have as yet no arithmetic properties; these we proceed now to assign, employing the method already used in the systems $\mathfrak{F}$ and R.

Our first step is to place $\mathfrak{R}$ in relation to R.

Let $\alpha = (a_1, a_2, \cdots)$

be an element of $\mathfrak{R}$. If there exists a rational number a, such that

$$\lim a_n = a,$$

we say $\alpha = a.$

61. 1. *Every number* a *of* R *lies in* $\mathfrak{R}$.

For, consider the sequence,

$$a + 1,\ a + \tfrac{1}{2},\ a + \tfrac{1}{3},\ \cdots$$

This sequence is regular, since

$$a_n - a_\nu = \frac{1}{n} - \frac{1}{\nu}.$$

The number $\alpha = (a + 1,\ a + \frac{1}{2},\ a + \frac{1}{3},\ \cdots)$ therefore lies in $\mathfrak{R}$.

On the other hand, $a_n \doteq a$.

Hence

$$\alpha = a.$$

2. Let $a_1, a_2, \cdots$ be any sequence of rational numbers, having 0 as limit; then

$$0 = (a_1, a_2, \cdots).$$

In particular,

$$\begin{aligned} 0 &= (1, \tfrac{1}{2}, \tfrac{1}{3}, \cdots) \\ &= (-1, -\tfrac{1}{2}, -\tfrac{1}{3}, \cdots) \\ &= (1, -\tfrac{1}{2}, \tfrac{1}{3}, -\tfrac{1}{4}, \cdots) \\ &= (0, 0, 0, \cdots). \end{aligned}$$

62. 1. We define now the terms *equal*, *greater than*, *less than*. The object of this is simply to *arrange* or *order* the elements of $\mathfrak{R}$.

Let

$$\alpha = (a_1, a_2, \cdots),\quad \beta = (b_1, b_2, \cdots).$$

We say

$$\alpha = \beta, \text{ when } \lim(a_n - b_n) = 0; \tag{1}$$

or, what is the same thing, when

$$\epsilon > 0,\ m,\quad |a_n - b_n| < \epsilon. \qquad n > m. \tag{2}$$

2. We say $\alpha > \beta$ when there exists a positive rational number r and an index m, such that

$$a_n - b_n > r. \qquad n > m. \tag{3}$$

We say similarly, $\alpha < \beta$, if

$$b_n - a_n > r, \qquad n > m. \tag{4}$$

or

$$a_n - b_n < -r. \tag{5}$$

3. Numbers of $\mathfrak{R}$ which are > 0 are called *positive*; those < 0 are *negative*.

63. It can be shown that from this definition of equality and inequality the usual properties of these terms can be deduced.

For example,

If $\alpha = \beta$, $\beta = \gamma$, *then* $\alpha = \gamma$.

For, setting

$$\alpha = (a_1, a_2, \cdots)$$

$$\beta = (b_1, b_2, \cdots),$$

$$\gamma = (c_1, c_2, \cdots),$$

we have

$$a_n - c_n = (a_n - b_n) + (b_n - c_n) \doteq 0,$$

since

$$a_n - b_n \doteq 0, \ b_n - c_n \doteq 0,$$

by hypothesis.

64. If $\alpha = (a_1, a_2, \cdots) = (a_1', a_2', \cdots)$, we say $(a_1, a_2, \cdots)$ and $(a_1', a_2', \cdots)$ are *different representations* of the same number α.

Every number α *in* $\mathfrak{R}$ *admits an infinity of representations.*

In fact, there are obviously an infinity of rational sequences

$$z_1, z_2, z_3, \cdots$$

having zero as limit.

Then

$$(a_1 + z_1, a_2 + z_2, \cdots)$$

represent an infinity of representations of α.

65. 1. We wish to apply the definition of 62 to the case that one of the members, say β, is a rational number b.

Let $\alpha = b$.

For $\beta = b$, we can take the representation

$$\beta = b = (b, b, \cdots). \qquad (1$$

Then 62, 1) requires that

$$\lim (a_n - b) = 0;$$

whence

$$\lim a_n = b.$$

Thus the definitions of 60 and 62 are in accord for this case.

2. *Let* $\alpha > b$.

Since $b_n = b$ by 1), the relation 62, 3) becomes here

$$a_n - b > r, \qquad n > m. \tag{2}$$

Let $\alpha < b$.

Then

$$b - a_n > r, \qquad n > m. \tag{3}$$

or

$$a_n - b < -r. \tag{4}$$

3. *If* $\alpha = (a_1, a_2, \cdots) > 0$, *there exists an index* m, *and two positive rational numbers* A, B, *such that*

$$A < a_n < B; \qquad n > m. \tag{5}$$

and conversely.

For, set $b = 0$, then 2) gives, replacing r by A,

$$a_n > A > 0. \qquad n > m. \tag{6}$$

On the other hand, 59 gives

$$|a_n| = a_n < B. \qquad n > m. \tag{7}$$

From 6) and 7), we have 5). The second half of the theorem is obvious, by 2.

4. Similarly, we have

If $\alpha = (a_1, a_2, \cdots) < 0$, *there exists an index* m, *and two negative rational numbers* $-A$, $-B$, *such that*

$$-A < a_n < -B; \qquad n > m.$$

and conversely.

5. From 3 and 4 we have

If $\alpha = (a_1, a_2, \cdots) \neq 0$, *there exists an index* m, *and two positive numbers* A, B, *such that*

$$A < |a_n| < B; \qquad n > m.$$

6. *In any number* $\alpha = (a_1, a_2, \cdots) \neq 0$, *the constituents* a_n *finally have one sign.*

This follows at once from 3 and 4.

66. 1. *Let* $\alpha = (a_1, a_2, \cdots)$.

If $$a_n \geqq a, \qquad n > m. \tag{1}$$

Then $$\alpha \geqq a. \tag{2}$$

For, suppose $\alpha < a$. Then, by 65, 2, there exists an $r > 0$, and an m, such that

$$a - a_n > r. \qquad n > m.$$

Hence

$$a > a_n + r > a_n;$$

and therefore

$$a_n < a,$$

which contradicts 1). Hence 2) holds.

2. Similarly, we show:

Let $\alpha = (a_1, a_2, \cdots)$.

If $$a_n \leqq a, \qquad n > m,$$

then $$\alpha \leq a.$$

67. 1. *If from the sequence*

$$a_1, \quad a_2, \quad a_3, \quad \cdots \tag{1}$$

which defines the number α, *we pick out a sequence*

$$a_{\iota_1}, \quad a_{\iota_2}, \quad a_{\iota_3}, \quad \cdots \tag{2}$$

where $\iota_1 < \iota_2 < \iota_3 \cdots$; *then also*

$$\alpha = (a_{\iota_1}, \ a_{\iota_2}, \ \cdots).$$

The sequence 2) *is regular.* For, since 1) is regular,

$$\epsilon > 0, \quad m, \quad |a_n - a_\nu| < \epsilon, \quad n, \nu > m.$$

But then

$$|a_{\iota_r} - a_{\iota_s}| < \epsilon, \quad r, s < \kappa, \quad \iota_\kappa > m.$$

Hence 2) is regular, and defines a number β.

We show now $\alpha = \beta$. Since 2) contains only a part of 1),

$$\iota_n \geqq n, \quad n = 1, 2, 3, \cdots$$

Since 1) is regular,

$$|a_n - a_{\iota_n}| < \epsilon. \qquad n > m.$$

Hence, by 62, 1, $$\alpha = \beta.$$

2. As corollary we have:

The number $\alpha = (a_1, a_2, \cdots)$ *is not altered, if we remove from or add to the numbers in the parenthesis, a finite number of rational numbers.*

3. We have also:

If in $$\alpha = (a_1, a_2, \cdots),\quad \beta = (b_1, b_2, \cdots)$$
$$a_n = b_n,\quad n > m;$$
then $$\alpha = \beta.$$

68. 1. *If* $\alpha = (a_1, a_2, \cdots) \neq 0$, *there cannot be an infinite number of constituents* $a_n = 0$.

For, say
$$a_{\iota_1} = a_{\iota_2} = a_{\iota_3} = \cdots = 0.$$
Then, by 67, 1,
$$\alpha = (a_{\iota_1},\ a_{\iota_2},\ \cdots).$$
But
$$(a_{\iota_1}, a_{\iota_2}, \cdots) = (0, 0, \cdots) = 0.$$
Hence $\alpha = 0$, which is a contradiction.

2. *If* $\alpha \neq 0$, *we can choose a representation* $(a_1, a_2, \cdots)$, *in which all the* $a_n \neq 0$.

For, let
$$\alpha = (a_1',\ a_2',\ \cdots) \tag*{(1}$$
be any representation of α. By 1, it contains but a finite number of zero. If we leave these zeros out of 1), we do not change the value of α, by 67, 2; and get thereby a representation of α, none of whose constituents are zero.

69. 1. Having ordered the elements of $\mathfrak{R}$, we proceed to define the *rational operations* upon them.

Addition.

Let
$$\alpha = (a_1, a_2, \cdots),\quad \beta = (b_1, b_2, \cdots)$$
be two elements of $\mathfrak{R}$, different or not; then
$$\alpha + \beta = (a_1 + b_1,\ a_2 + b_2, \cdots). \tag*{(1}$$

To justify this definition of addition, we show first that

$$a_1 + b_1,\quad a_2 + b_2,\quad \cdots \qquad\qquad (2$$

is a regular sequence.

Since $a_1, a_2, \cdots$ is a regular sequence, we have

$$\epsilon > 0,\quad m,\quad |a_n - a_\nu| < \epsilon/2,\quad n, \nu > m. \qquad\qquad (3$$

Since $b_1, b_2, \cdots$ is regular, we have

$$\epsilon < 0,\quad m,\quad |b_n - b_\nu| < \epsilon/2,\quad n, \nu > m. \qquad\qquad (4$$

Evidently we can take m so large that 3), 4) hold for the same m.

Now

$$\begin{aligned} |(a_n + b_n) - (a_\nu + b_\nu)| &= |(a_n - a_\nu) + (b_n - b_\nu)| \\ &< |a_n - a_\nu| + |b_n - b_\nu|, \text{ by 37, 3) ;} \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \text{ by 3), 4).} \end{aligned}$$

Thus 2) is regular, and defines a number.

2. We show next that, if α, β are rational numbers, say $\alpha = a$, $\beta = b$; then $\alpha + \beta$, as defined by 1), is $a + b$.

Since α is a rational number a,

$$\lim a_n = a, \text{ by 60.}$$

Similarly,

$$\lim b_n = b.$$

But

$$\lim (a_n + b_n) = \lim a_n + \lim b_n = a + b, \text{ by 49.}$$

Thus by 60,

$$(a_1 + b_1,\ a_2 + b_2,\ \cdots) = a + b.$$

Hence by 1),

$$\alpha + \beta = a + b.$$

70. 1. *If* $\beta > \gamma$, *then* $\alpha + \beta > \alpha + \gamma$.

Let $\gamma = (c_1, c_2, \cdots)$. Since $\beta > \gamma$, there exists, by 62, 2, a positive rational number r, such that

$$b_n > c_n + r. \qquad n > m.$$

Hence, adding a_n,

$$a_n + b_n > a_n + c_n + r.$$

$$\therefore\ (a_n + b_n) - (a_n + c_n) > r.$$

Hence, by 62, 2,

$$\alpha + \beta > \alpha + \gamma.$$

2. From 1, we conclude, as in 26, that

If $\alpha+\beta=\alpha+\gamma$, *then* $\beta=\gamma$.

71. 1. *Subtraction.*

This is the inverse of addition; we define it as we did in $\mathfrak{F}$ and *R*, viz.: The result of subtracting β from α is the number or numbers ξ, in $\mathfrak{R}$, which satisfy

$$\alpha=\beta+\xi. \tag{1}$$

There is at most one number ξ.

For, suppose $\alpha=\beta+\eta$. (2

Then 1), 2) give, by 63,

$$\beta+\xi=\beta+\eta.$$

Hence, by 70, 2, $\eta=\xi$.

To show that 1) admits *one* solution, we prove just as in 69, 1, that

$$a_1-b_1,\ a_2-b_2,\ \cdots$$

is a regular sequence, and thus defines a number

$$\xi=(a_1-b_1,\ a_2-b_2,\ \cdots).$$

If we put this value of ξ in 1), the equation is satisfied.

For,
$$\begin{aligned}\beta+\xi&=(b_1,\ b_2,\ \cdots)+(a_1-b_1,\ a_2-b_2,\ \cdots)\\&=(b_1+a_1-b_1,\ b_2+a_2-b_2,\ \cdots),\text{ by 69, 1)}\\&=(a_1,\ a_2,\ \cdots)=\alpha.\end{aligned}$$

2. Thus subtraction is always possible in $\mathfrak{R}$, and is unique. The result of subtracting β from α we represent by $\alpha-\beta$; we have then

$$\alpha-\beta=(a_1-b_1,\ a_2-b_2,\ \cdots).$$

3. We represent $0-\alpha$ by $-\alpha$.

Evidently,
$$-\alpha=(-a_1,\ -a_2,\ -a_3,\ \cdots).$$

We observe that
$$\begin{aligned}\alpha+(-\alpha)&=0;\\\alpha+(-\beta)&=\alpha-\beta;\\-(-\alpha)&=\alpha.\end{aligned}$$

72. 1. *If α is positive, $-\alpha$ is negative; and if α is negative, $-\alpha$ is positive.*

For, if $\alpha=(a_1, a_2, \cdots)>0$, we have, by 65, 3,

$$a_n > A > 0. \qquad n>m.$$

Now

$$-\alpha=(-a_1, -a_2, \cdots), \text{ by 71, 3.}$$

Hence, by 1),

$$-a_n < -A < 0. \qquad n>m.$$

Hence, by 65, 4,

$$-\alpha<0,$$

which proves the first part of the theorem. The second part is proved similarly.

2. *All the numbers of $\mathfrak{R} \neq 0$ are of the form α or $-\alpha$, where α is a positive number.*

Let β be a number $\neq 0$. We need to consider only the case that β is negative.

By 71, 3,

$$\beta=-(-\beta);$$

and by 1), $-\beta$ is positive.

73. 1. *Multiplication.*

The product of α by β we define by

$$\alpha\beta=(a_1b_1, \ a_2b_2, \ \cdots). \tag{1}$$

We have to show that

$$a_1b_1, \ a_2b_2, \ \cdots \tag{2}$$

is a regular sequence.

Let ϵ be a positive rational number, small at pleasure.

Then, by 59, there exists a positive M, such that,

$$|a_n|, \ |b_n|<M. \qquad n>m. \tag{3}$$

Also, since the sequences $\{a_n\}$, $\{b_n\}$ are regular, we can suppose m in 3) is taken so large that

$$|a_n-a_\nu|, \ |b_n-b_\nu|<\frac{\epsilon}{2M}. \qquad n, \nu>m. \tag{4}$$

Now,
$$d_n = a_n b_n - a_\nu b_\nu = a_n(b_n - b_\nu) + b_\nu(a_n - a_\nu).$$

$$\therefore\ |d_n| \leqq |a_n||b_n - b_\nu| + |b_\nu||a_n - a_\nu|, \text{ by 37,}$$

$$< M \cdot \frac{\epsilon}{2M} + M\frac{\epsilon}{2M}, \text{ by 3) and 4).}$$

$$\therefore\ |d_n| < \epsilon,$$

and 2) is regular.

2. If α, β are rational, say $\alpha = a$, $\beta = b$, we show that $\alpha\beta$ as defined in 1) is ab.

For, since α and β are rational,

$$\lim a_n = a,\ \lim b_n = b, \text{ by 60.}$$

But then, by 50,

$$\lim a_n b_n = \lim a_n \lim b_n = ab,$$

which states that $\alpha\beta = ab$.

74. 1. The formal laws for addition and multiplication are readily proved. We illustrate this by establishing the *associative law of multiplication.*

We wish to show that

$$\alpha \cdot \beta\gamma = \alpha\beta \cdot \gamma. \tag*{(1}$$

We have, by 73, 1),

$$\beta\gamma = (b_1c_1,\ b_2c_2,\ \cdots).$$

Hence

$$\alpha \cdot \beta\gamma = (a_1,\ a_2,\ \cdots)(b_1c_1,\ b_2c_2,\ \cdots)$$

$$= (a_1 \cdot b_1c_1,\ a_2 \cdot b_2c_2,\ \cdots). \tag*{(2}$$

Similarly,

$$\alpha\beta \cdot \gamma = (a_1b_1 \cdot c_1,\ a_2b_2 \cdot c_2,\ \cdots). \tag*{(3}$$

Since multiplication is associative in R, the two numbers represented by 2), 3) are identical, which proves 1).

2. As a consequence of the associative law, we have, m, n being positive integers,

$$\alpha^m\alpha^n = \alpha^{m+n},$$

which expresses *the addition theorem for integral positive exponents.*

75. 1. The properties of positions relating to greater than, less than, are readily established for numbers in $\mathfrak{A}$.

If $\alpha > \beta$, and $\gamma > 0$, then $\alpha\gamma > \beta\gamma$.

For, since $\alpha > \beta$, we have, by 62, 2,

$$a_n - b_n > r > 0. \qquad n > m.$$

Since $\gamma > 0$, there exists a positive rational number e, by 65, 3, such that

$$c_n > e. \qquad n > m.$$

By taking m sufficiently large, we may take the same m in both these inequalities.

They give

$$a_n c_n - b_n c_n > er > 0.$$

Then, by 62, 2,

$$\alpha\gamma > \beta\gamma.$$

2. From 1 follows:

If $\alpha > \beta > 0$,

then $$\alpha^n > \beta^n. \qquad n \text{ positive integer.}$$

3. From 2 we conclude:

If $\alpha, \beta > 0$, and $\alpha^n = \beta^n$, n being a positive integer, then

$$\alpha = \beta.$$

4. *If $0 < \alpha < 1$, then $\alpha^n < \alpha$.*

For, from

$$\alpha < 1,$$

we have

$$\alpha^2 < \alpha.$$

Also

$$\alpha^3 < \alpha^2.$$

Hence

$$\alpha^3 < \alpha.$$

Hence, in general,

$$\alpha^n < \alpha.$$

76. 1. *Rule of signs: The product of two positive or two negative numbers in $\mathfrak{A}$ is positive. The product of a positive and a negative number is negative.*

Let $\alpha > 0$, $\beta > 0$; *then* $\alpha\beta > 0$.

By 65, 3, there exist two positive numbers A, B, and an index m, such that

$$a_n > A, \qquad b_n > B. \qquad n > m.$$

Hence

$$a_n b_n > AB > 0.$$

Thus

$$\alpha\beta = (a_1 b_1, a_2 b_2, \cdots) > 0, \text{ by 66.}$$

Let $\alpha > 0$, $\beta < 0$; *then* $\alpha\beta < 0$

For, by 65, 3, 4,

$$a_n > A, \qquad b_n < -B. \qquad n > m.$$

$$\therefore \ a_n b_n < -AB < 0.$$

Thus, by 66,

$$\alpha\beta < 0.$$

In a precisely similar manner, we can treat the other cases.

77. 1. *The product of any two numbers in* $\mathfrak{R}$ *vanishes when, and only when, one of the factors is zero.*

In the product $\alpha\beta$, *suppose* $\alpha = 0$; *then* $\alpha\beta = 0$.

Then

$$\alpha = (0, 0, 0, \cdots), \quad \beta = (b_1, b_2, \cdots).$$

$$\therefore \ \alpha\beta = (0 \cdot b_1, 0 \cdot b_2, \cdots) = (0, 0, \cdots) = 0.$$

Conversely, if $\alpha\beta = 0$, either α or $\beta = 0$.

This is proved, as in 25.

2. *If* $\alpha \neq 0$, *and* $\alpha\beta = \alpha\gamma$, *then* $\beta = \gamma$.

Proof same as that for 26, 3).

78. 1. *Division.*

The quotient of α by β is the number or numbers ξ, in $\mathfrak{R}$, which satisfy

$$\alpha = \beta\xi. \tag{1}$$

There are two cases, according as $\beta = 0$, or $\neq 0$.

Case I; $\beta \neq 0$.

Since $\beta \neq 0$, we may suppose, by 68, 2, that in

$$\beta = (b_1, b_2, \cdots),$$

all $b_n \neq 0$. To find a solution of 1), consider the sequence

$$\frac{a_1}{b_1}, \frac{a_2}{b_2}, \cdots \tag{2}$$

It is regular. For,

$$d_{n,\nu} = \frac{a_n}{b_n} - \frac{a_\nu}{b_\nu} = \frac{a_n b_\nu - a_\nu b_n}{b_n b_\nu} = \frac{a_n(b_\nu - b_n) - b_n(a_\nu - a_n)}{b_n b_\nu}.$$

Hence

$$|d_{n,\nu}| \leq \frac{|a_n||b_\nu - b_n| + |b_n||a_\nu - a_n|}{|b_n||b_\nu|}. \tag{3}$$

By 59,

$$|a_n| < M. \qquad n > m. \tag{4}$$

By 65, 5, we have

$$A < |b_n| < B. \qquad n > m. \tag{5}$$

By taking m sufficiently large we may suppose it to have the same value in 4), 5).

Then 4), 5) gives in 3),

$$|d_{n,\nu}| < \frac{M|b_\nu - b_n| + B|a_\nu - a_n|}{A^2}. \tag{6}$$

Since the sequences $\{a_n\}$, $\{b_n\}$ are regular, we may now suppose m taken so large that also

$$|a_\nu - a_n| < \frac{\epsilon A^2}{2B}, \quad |b_\nu - b_n| < \frac{\epsilon A^2}{2M}. \qquad n, \nu > m.$$

Then 6) gives

$$|d_{n,\nu}| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Since 2) is regular, it defines a number

$$\xi = \left(\frac{a_1}{b_1}, \frac{a_2}{b_2}, \cdots\right).$$

Since

$$\beta \cdot \xi = (b_1, b_2, \cdots)\left(\frac{a_1}{b_1}, \frac{a_2}{b_2}, \cdots\right)$$

$$= (a_1, a_2, \cdots) = \alpha, \text{ by 73, 1)},$$

ξ satisfies 1).

That this is the only solution of 1) follows as in 30, 1, from 77, 2.

Case II; $\beta = 0$.

We can reason precisely as we did in 30, 2. Hence, when the divisor $\beta = 0$, division is either impossible or entirely indeterminate. For this reason division by 0 is excluded.

2. We have thus this result: in the system $\mathfrak{R}$, division is always possible and unique, except when the divisor is 0, when division is not permissible.

3. The result of dividing α by β, we represent by α/β and have therefore

$$\frac{\alpha}{\beta} = \left(\frac{a_1}{b_1}, \frac{a_2}{b_2}, \cdots\right).$$

Since

$$1 = (1, 1, 1, \cdots),$$

$$\frac{1}{\beta} = \left(\frac{1}{b_1}, \frac{1}{b_2}, \cdots\right).$$

This is called the reciprocal of β.

79. 1. The system $\mathfrak{R}$ is now completely defined; its elements have been ordered, and the four rational operations upon them have been defined. As a perfect analogy exists between the systems R and $\mathfrak{R}$, we are justified in calling the elements of $\mathfrak{R}$ numbers. In the future, when speaking of numbers, without further predicate, we shall mean the numbers in $\mathfrak{R}$. As already stated, they are called real numbers.

2. In the ϵ, m test, given in 57, we were obliged at that stage to take ϵ rational. This is now quite unnecessary, and we shall therefore, in the future, suppose ϵ is any positive number in $\mathfrak{R}$, small at pleasure.

80. 1. We have shown in 61 that $\mathfrak{R}$ contains all the numbers of R; but we have not shown that it contains other numbers.

To this end, we show that there is a number α which satisfies

$$x^2 = 2. \qquad (1$$

This is easily done. For in 54 we determined a rational sequence

$$a_1 = 1,\ a_2 = 1.4,\ a_3 = 1.41,\ \cdots \qquad (2$$

such that

$$\lim a_n^2 = 2. \qquad (3$$

The sequence 2) is regular by 58, 1.

Hence

$$\alpha = (a_1,\ a_2,\ \cdots)$$

is a number in $\mathfrak{R}$.

But, by 73, 1),

$$\alpha^2 = (a_1^2,\ a_2^2,\ \cdots).$$

Hence 3) shows, by 60, that

$$\alpha^2 = 2.$$

Hence α is a solution of 1).

2. As we saw in 52 that α is not rational, we have shown there is at least one number in $\mathfrak{R}$ not in R.

But the reasoning we have just applied to $\sqrt{2}$ applies equally to $\sqrt[n]{a}$, when this latter is not rational. There are thus an infinity of numbers in $\mathfrak{R}$ not in R.

Some Properties of $\mathfrak{R}$

81. *If* $\alpha > 0$, *there are an infinity of positive rational numbers* $< \alpha$, *and also an infinity of rational numbers* $> \alpha$.

If α is rational, the theorem is obviously true by 33.

Let

$$\alpha = (a_1,\ a_2,\ \cdots).$$

Then, by 65, 3,

$$0 < A < a_n < B. \qquad n > m. \tag{1}$$

But from

$$a_n > A,$$

we have, by 66, 1,

$$\alpha \geqq A.$$

Since there are an infinity of rational numbers between 0 and the positive rational number A, the first half of the theorem is established.

Using the other part of the inequality 1), we prove similarly the rest of the theorem.

82. *Between* α, β, *lie an infinity of rational numbers.*

For, let $\alpha < \beta$; then, by 81, there exist positive rational numbers A, B, d, such that

$$A < \alpha,\ B > \beta,\ d < \beta - \alpha.$$

Let
$$D = B - A;$$
we can, by 34, 2, determine the positive integer n so great that
$$\frac{D}{n} < d.$$

Then, at least one of the numbers
$$A + \frac{D}{n},\ A + 2\frac{D}{n},\ \cdots A + (n-1)\frac{D}{n}$$
falls between α and β.

83. 1. *The system $\mathfrak{R}$ is Archimedian;* i.e. *for each pair of positive numbers $\alpha < \beta$ there exists a positive integer n, such that $n\alpha > \beta$.*

For, by 81, there exist positive rational numbers
$$a < \alpha,\ b > \beta.$$

Since the system R is Archimedian [34, 1], there exists an integer n, such that
$$na > b.$$

But
$$n\alpha > na, \text{ and } b > \beta.$$

Hence
$$n\alpha > \beta.$$

2. *For any pair of positive numbers $\alpha < \beta$ there exists a positive n such that*
$$\frac{\beta}{n} < \alpha.$$

Proof, as in 34, 2.

84. *Between α and β, $\alpha < \beta$, lie an infinity of irrational numbers.*

That irrational numbers exist, we have shown in 80.

Let i be an irrational number, r a rational number, and n a positive integer.

Then
$$j = \frac{i}{n},\ k = i + r$$
are irrational.

For, if j were rational, $i = nj$ is rational. This is a contradiction.

Similarly, if k were rational, $i = k - r$ is rational, which is a contradiction.

This established, suppose first that α is rational and positive. Let i be any positive irrational number.

Then, by 83, 2, we can take n so large that

$$\frac{i}{n} < \beta - \alpha.$$

But then

$$\alpha < \alpha + \frac{i}{n} < \beta;$$

and

$$\alpha + \frac{i}{n}$$

is irrational.

Suppose now that α is irrational and positive.

By 81, there exists a positive rational number r, such that

$$0 < r < \beta - \alpha.$$

Then

$$\alpha < \alpha + r < \beta;$$

and

$$\alpha + r$$

is irrational.

The cases when α, β are one or both negative are now easily treated.

85. *The system* $\mathfrak{R}$ *is dense*, i.e. *between any two numbers of* $\mathfrak{R}$ *lie an infinity of numbers.*

This follows at once from 82 or 84.

Numerical Values and Inequalities

86. We have seen, 72, 2, that any number $\alpha \neq 0$ can be written

$$\alpha = \pm \alpha_0,$$

where α_0 is a positive number.

We define now, as in 36, *the numerical or absolute value of* α *is* $+\alpha_0$, and denote it by

$$|\alpha|.$$

Then by definition

$$|\alpha| = \alpha_0.$$

We set also

$$|0| = 0.$$

87. 1. We have now the following fundamental relations :

$$|\alpha| = |-\alpha|; \quad (1$$

$$|\alpha - \beta| = |\beta - \alpha|; \quad (2$$

$$|\alpha \pm \beta| \overline{\overline{<}} |\alpha| + |\beta|; \quad (3$$

$$|\alpha \pm \beta| \overline{\overline{>}} ||\alpha| - |\beta||; \quad (4$$

$$|\alpha\beta| = |\alpha| \cdot |\beta|; \quad (5$$

$$\left|\frac{\alpha}{\beta}\right| = \frac{|\alpha|}{|\beta|}; \qquad \beta \neq 0. \quad (6$$

$$|\alpha_1 \pm \alpha_2 \cdots \pm \alpha_m| \leq |\alpha_1| + \cdots + |\alpha_m|; \quad (7$$

$$|\alpha_1 \cdot \alpha_2 \cdots \alpha_m| = |\alpha_1| \cdot |\alpha_2| \cdots |\alpha_m|. \quad (8$$

2. *From*

$$|\alpha| < A,$$

follows

$$-A < \alpha < A;$$

and conversely.

3. *From*

$$|\alpha - \beta| < A, \quad |\beta - \gamma| < B,$$

follows

$$|\alpha - \gamma| < A + B;$$

or if $A = B$,

$$|\alpha - \gamma| < 2A.$$

4. As the demonstration of these relations is exactly the same as in 37, 38, we do not need to repeat it.

5. *If we know of two numbers* α, β, *that* $|\alpha - \beta| < \epsilon$ *however small* $\epsilon > 0$ *is taken; then*

$$\alpha = \beta.$$

The demonstration is the same as in **46**.

88. *If* $\alpha = (a_1, a_2, \cdots)$, *then*

$$|\alpha| = (|a_1|, |a_2|, \cdots).$$

Since the sequence $\qquad a_1, a_2, \cdots$

defines a number, it is regular.

Hence $\qquad \epsilon > 0, m, \quad |a_n - a_\nu| < \epsilon. \qquad n, \nu > m.$

From this we conclude that the sequence

$$|a_1|,\ |a_2|,\ \cdots \tag{1}$$

is regular.

For, by 87, 4),
$$\big|\,|a_n|-|a_\nu|\,\big| \leq |a_n - a_\nu|.$$

Hence 1) defines a number.

Set
$$\beta = (|a_1|,\ |a_2|,\ \cdots).$$

To show that $\beta = |\alpha|$.

First suppose $\alpha = 0$.

Then
$$\lim a_n = 0.$$

Hence
$$\lim |a_n| = 0.$$

Therefore
$$\beta = 0, \text{ and } \beta = |\alpha|.$$

Suppose $\alpha \neq 0$. Then, by 65, 6, the constituents a_n of α are of one sign, for $n > m$.

If $\alpha > 0$,
$$a_n = |a_n|. \qquad n > m.$$

Hence
$$\beta = (|a_1|,\ |a_2|,\ \cdots,\ |a_m|,\ a_{m+1},\ a_{m+2},\ \cdots)$$
$$= \alpha = |\alpha|, \text{ by 67, 3.}$$

If $\alpha < 0$,
$$a_n = -|a_n|. \qquad n > m.$$

Hence
$$\beta = (|a_1|,\ \cdots,\ |a_m|,\ -a_{m+1},\ -a_{m+2},\ \cdots)$$
$$= -(-|a_1|,\ \cdots,\ -|a_m|,\ a_{m+1},\ a_{m+2},\ \cdots)$$
$$= -\alpha, \text{ by 67, 3}$$
$$= |\alpha|.$$

89. In the following articles we give certain equalities and inequalities which are often useful.

Let $0 < \alpha < 1$; *then*
$$\frac{1}{1+\alpha} > 1 - \alpha, \tag{1}$$
$$\frac{1}{1-\alpha} > 1 + \alpha. \tag{2}$$

To prove 1), let us suppose the contrary, viz.:

$$\frac{1}{1+\alpha} \overline{\gtrless} 1-\alpha.$$

Clearing of fractions,

$$1 \leq 1-\alpha^2, \text{ or } \alpha^2 \leq 0,$$

which is a contradiction.

Similarly, we may prove 2).

90. 1. *Let* $\alpha_1, \alpha_2, \cdots, \alpha_m > 0$ *and* $P_m = (1+\alpha_1)(1+\alpha_2)\cdots(1+\alpha_m)$. *Then*

$$P_m > 1 + (\alpha_1 + \cdots + \alpha_m). \qquad m > 1.$$

$$P_m > 1 + (\alpha_1 + \cdots + \alpha_m) + (\alpha_1\alpha_2 + \alpha_1\alpha_3 + \cdots + \alpha_{m-1}\alpha_m). \quad m > 2.$$

In fact,

$$P_2 = (1+\alpha_1)(1+\alpha_2) = 1 + (\alpha_1+\alpha_2) + \alpha_1\alpha_2 > 1 + (\alpha_1 + \alpha_2);$$

$$\begin{aligned} P_3 = P_2(1+\alpha_3) &= 1 + (\alpha_1+\alpha_2+\alpha_3) + \alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3 + \alpha_1\alpha_2\alpha_3 \\ &> 1 + (\alpha_1+\alpha_2+\alpha_3) + \alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3 \\ &> 1 + (\alpha_1+\alpha_2+\alpha_3). \end{aligned}$$

In this way we can continue.

2. Similarly, we can prove:

Let $0 < \alpha_1, \alpha_2, \cdots \alpha_m < 1$, *and*

$$Q_m = (1-\alpha_1)(1-\alpha_2)\cdots(1-\alpha_m).$$

Then $Q_m > 1 - (\alpha_1 + \cdots + \alpha_m). \qquad m > 1.$

$$< 1 - (\alpha_1 + \cdots + \alpha_m) + (\alpha_1\alpha_2 + \alpha_1\alpha_3 + \cdots + \alpha_{m-1}\alpha_m). \quad m > 2.$$

91. The demonstration of the following identities is obvious:

$$\frac{1}{1-\alpha} = 1 + \alpha + \alpha^2 + \cdots + \alpha^{n-1} + \frac{\alpha^n}{1-\alpha}. \tag*{(1}$$

$$\frac{\alpha+\epsilon}{\beta+\delta} = \frac{\alpha}{\beta} + \frac{\beta\epsilon - \alpha\delta}{\beta(\beta+\delta)}. \tag*{(2}$$

92. *Let* $|\delta_1|, |\delta_2| < \delta$, and $\beta \neq 0$; *let*

$$\rho = \frac{\alpha + \delta_1}{\beta + \delta_2}.$$

Let $\epsilon > 0$ *be small at pleasure; we can take* $\delta > 0$ *so small that*

$$\rho = \frac{\alpha}{\beta} + \sigma. \qquad |\sigma| < \epsilon. \tag{1}$$

For, by 91, 2,

$$\sigma = \frac{\delta_1\beta - \delta_2\alpha}{\beta(\beta + \delta_2)}.$$

Hence

$$|\sigma| \le \frac{\delta(|\alpha| + |\beta|)}{|\beta| \cdot \big||\beta| - \delta\big|}, \quad \text{by 37,} \tag{2}$$

taking δ so small that

$$|\beta| - \delta > 0.$$

Let

$$|\alpha|, \ |\beta| < g,$$

$$|\beta|, \ |\beta| - \delta > h. \qquad h > 0.$$

Then 2) gives

$$|\sigma| < \frac{2\,\delta g}{h^2}.$$

Hence, if we take

$$\delta < \frac{h^2\epsilon}{2\,g},$$

we have 1).

93. *Let* $\alpha_1, \alpha_2, \cdots \alpha_n$ *be* n *arbitrary numbers.*

Let $\beta_1 \cdots \beta_n;\ \gamma_1 \cdots \gamma_n > 0.$

If

$$L \le \frac{\alpha_1}{\beta_1}, \ \frac{\alpha_2}{\beta_2}, \ \cdots \ \frac{\alpha_n}{\beta_n} \le G,$$

then

$$L \le \frac{\gamma_1\alpha_1 + \cdots + \gamma_n\alpha_n}{\gamma_1\beta_1 + \cdots + \gamma_n\beta_n} \le G.$$

For, from

$$\frac{\alpha_1}{\beta_1} \ge L, \ \cdots, \ \frac{\alpha_n}{\beta_n} \ge L,$$

we have

$$\gamma_1\alpha_1 \ge \beta_1\gamma_1 L, \ \cdots \ \gamma_n\alpha_n \ge \beta_n\gamma_n L.$$

Adding,

$$\gamma_1\alpha_1 + \cdots + \gamma_n\alpha_n \ge L(\beta_1\gamma_1 + \cdots + \beta_n\gamma_n),$$

which gives the first half of 1). The rest of 1) follows similarly.

94. 1. *Let* $\alpha > \beta \geqq 0$; *and* $n > 1$, *a positive integer.*
Then

$$n(\alpha - \beta)\beta^{n-1} < \alpha^n - \beta^n < n(\alpha - \beta)\alpha^{n-1}. \qquad (1$$

For, by direct multiplication, we verify

$$\alpha^n - \beta^n = (\alpha - \beta)(\alpha^{n-1} + \alpha^{n-2}\beta + \alpha^{n-3}\beta^2 + \cdots + \beta^{n-1}). \qquad (2$$

In the second parenthesis, replace α by β. Then, since $\alpha > \beta$,

$$\alpha^n - \beta^n > (\alpha - \beta)(\beta^{n-1} + \beta^{n-1} + \cdots n \text{ terms}),$$

or

$$\alpha^n - \beta^n > n(\alpha - \beta)\beta^{n-1},$$

which is a part of 1).

If in 2) we replace β by α, we get the other half of 1).

2. In 1), set $\alpha = 1 + \delta$, $\delta > 0$, $\beta = 1$, we get

$$(1 + \delta)^n > 1 + n\delta.$$

If we set $\alpha = 1$, $\beta = 1 - \delta$, 1) gives

$$(1 - \delta)^n > 1 - n\delta.$$

We have thus

$$(1 + \alpha)^n > 1 + n\alpha, \ \alpha \neq 0 \text{ and } \geqq -1, \ n \text{ positive integer.} \qquad (3$$

3. We observe that 1) can be written

$$\alpha^n > \beta^{n-1}[\beta + n(\alpha - \beta)], \qquad (4$$

$$\beta^n > \alpha^{n-1}[\alpha - n(\alpha - \beta)]. \qquad (5$$

95. Let $\alpha_1 \cdots \alpha_n$ be any n numbers.

$$A_n = \frac{\alpha_1 + \cdots + \alpha_n}{n}$$

is called their *arithmetic mean.*

Let $\alpha_1 \cdots \alpha_n$ *be positive, and* $P_n = \alpha_1 \cdot \alpha_2 \cdots \alpha_n$.
Then $P_n < A_n^n$, *unless the* α*'s are all equal, when* $P_n = A_n^n$.
If $\alpha_1 = \alpha_2 = \cdots = \alpha_n$, $A_n = \alpha_1$ and $P_n = a_1^n$.

Hence

$$P_n = A_n^n.$$

Suppose now the α's are not all equal.

Let $n = 2$.

We have
$$\alpha_1\alpha_2 = \left(\frac{\alpha_1+\alpha_2}{2}\right)^2 - \left(\frac{\alpha_1-\alpha_2}{2}\right)^2 < \left(\frac{\alpha_1+\alpha_2}{2}\right)^2.$$

Hence
$$P_2 < A_2^2.$$

Let $n = 2^m$.

Since the α's are not all equal, at least two of them, say α_1, α_2, are unequal.

Then
$$\alpha_1\alpha_2 < \left(\frac{\alpha_1+\alpha_2}{2}\right)^2,$$
and
$$\alpha_3\alpha_4 \leq \left(\frac{\alpha_3+\alpha_4}{2}\right)^2.$$

Hence
$$\alpha_1\alpha_2\alpha_3\alpha_4 < \left(\frac{\alpha_1+\alpha_2}{2}\right)^2\left(\frac{\alpha_3+\alpha_4}{2}\right)^2. \qquad (1$$

On the other hand, applying our theorem to
$$\frac{\alpha_1+\alpha_2}{2},\quad \frac{\alpha_3+\alpha_4}{2},$$
we have
$$\frac{\alpha_1+\alpha_2}{2}\cdot\frac{\alpha_3+\alpha_4}{2} \leq \left(\frac{\alpha_1+\alpha_2+\alpha_3+\alpha_4}{4}\right)^2. \qquad (2$$

Hence 1) and 2) give
$$P_4 < A_4^4.$$

In the same way, we may continue for any power of 2.

Let $2^{m-1} < n < 2^m$. Set $\mu = 2^m$, $2^m - n = \nu$.

We have, by the preceding case,
$$\alpha_1\alpha_2\cdots\alpha_\mu < \left(\frac{\alpha_1+\cdots+\alpha_\mu}{\mu}\right)^\mu. \qquad (3$$

Set here
$$\alpha_{n+1} = \alpha_{n+2} = \cdots = \alpha_\mu = A_n.$$

Then 3) gives
$$P_n A_n^\nu < \left(\frac{\alpha_1+\cdots+\alpha_n+\nu A_n}{\mu}\right)^\mu = A_n^\mu, \qquad (4$$
since
$$\frac{\alpha_1+\cdots+\alpha_n+\nu A_n}{\mu} = \frac{nA_n+\nu A_n}{\mu} = \frac{\mu A_n}{\mu} = A_n.$$

Dividing in 4) by A_n^ν, we get
$$P_n < A_n^n.$$

96. From algebra we have the *Binomial Theorem*,

$$(\alpha+\beta)^n=\alpha^n+n\alpha^{n-1}\beta+\frac{n\cdot n-1}{1\cdot 2}\alpha^{n-2}\beta^2+\frac{n\cdot n-1\cdot n-2}{1\cdot 2\cdot 3}\alpha^{n-3}\beta^3$$

$$+\cdots+\frac{n\cdot n-1}{1\cdot 2}\alpha^2\beta^{n-2}+n\alpha\beta^{n-1}+\beta^n, \quad (1$$

where n is a positive integer.

The *binomial coefficients*

$$\frac{n\cdot n-1\cdot n-2\cdots n-m+1}{1\cdot 2\cdots m},$$

we denote by

$$\binom{n}{m}.$$

We have obviously,

$$\binom{n}{n}=1.$$

$$\binom{n}{m}+\binom{n}{m-1}=\binom{n+1}{m}.$$

If we set $\alpha=\beta=1$ in 1), we get

$$2^n=1+\binom{n}{1}+\binom{n}{2}+\cdots+\binom{n}{n-1}+\binom{n}{n}.$$

If we set $\alpha=1$, $\beta=-1$ in 1), we get

$$0=1-\binom{n}{1}+\binom{n}{2}-\cdots+(-1)^n\binom{n}{n}.$$

It is often convenient to set

$$\binom{n}{0}=1,$$

and

$$\binom{n}{m}=0, \text{ if } m>n.$$

Limits

97. We extend now the terms *sequence*, *regular sequence*, *limit*, etc., to numbers in $\mathfrak{R}$. This is done at once; for the definitions given in 40, 42, and 57 may be extended to $\mathfrak{R}$, by simply replacing the term *rational number* by *number in* $\mathfrak{R}$.

For example, the sequence of numbers in $\mathfrak{R}$

$$\alpha_1, \quad \alpha_2, \quad \alpha_3, \quad \cdots \tag*{(1}$$

is *regular* when, for each positive ϵ (not necessarily a rational number now) there exists an index m, such that

$$|\alpha_n - \alpha_\nu| < \epsilon,$$

for every pair of indices $n, \nu > m$.

Or in *abbreviated form*, when

$$\epsilon > 0, \quad m, \quad |\alpha_n - \alpha_\nu| < \epsilon. \quad n, \nu > m. \tag*{(2}$$

This definition, we see, is perfectly analogous to that given in 57, 1 for regular rational sequences. Evidently the reasoning of 57, 3, 4, can be applied to the sequence 1). Thus the ϵ, m *test* given in 2 may also be stated in the form:

$$\epsilon > 0, \quad m, \quad |\alpha_n - \alpha_m| < \epsilon, \quad n > m. \tag*{(3}$$

Similarly, λ is the *limit of the sequence*

$$\alpha_1, \quad \alpha_2, \quad \alpha_3, \quad \cdots$$

when

$$\epsilon > 0, \quad m, \quad |\lambda - \alpha_n| < \epsilon, \quad n > m. \tag*{(4}$$

As before, we write

$$\lambda = \lim_{n=\infty} \alpha_n, \quad \text{or} \quad \lambda = \lim \alpha_n.$$

We say also α_n *converges to* λ or *approaches* λ as limit.

This may be indicated by the notation

$$\alpha_n \doteq \lambda.$$

98. *Let* $\lim \alpha_n = \alpha$ *and* $\lim \beta_n = \beta$.

Then
$$\lim (\alpha_n \pm \beta_n) = \alpha \pm \beta; \tag*{(1}$$

$$\lim \alpha_n \beta_n = \alpha\beta. \tag*{(2}$$

If $\beta, \beta_1, \beta_2, \cdots \neq 0$, *we have also*

$$\lim \frac{\alpha_n}{\beta_n} = \frac{\alpha}{\beta}. \tag*{(3}$$

The demonstration is precisely similar to those of 49, 50, 51; and thus does not need to be repeated here.

99. We prove now the important theorem:

Let $\alpha = (a_1, a_2, \cdots)$, *the a's rational; then* $\lim a_n = \alpha$.

We must show that

$$\epsilon > 0, \quad m, \quad |\alpha - a_n| < \epsilon, \quad n > m. \tag*{(1}$$

Since the sequence

$$a_1, \quad a_2, \quad a_3, \quad \cdots$$

is regular, we have

$$\sigma > 0, \quad m, \quad |a_n - a_\nu| < \sigma, \quad n, \nu > m. \tag*{(2}$$

Now we can write, by 60,

$$a_n = (a_n, a_n, a_n, \cdots).$$

Hence, by 71, supposing n to be fixed for the moment,

$$\alpha - a_n = (a_1 - a_n, \ a_2 - a_n, \ a_3 - a_n, \ \cdots).$$

By 88,

$$|\alpha - a_n| = (|a_1 - a_n|, \ |a_2 - a_n|, \ \cdots).$$

Hence, by 2) and 66, 2,

$$|\alpha - a_n| \leq \sigma.$$

Thus if we take $\sigma < \epsilon$, we have 1).

100. *If a sequence*

$$A = \alpha_1, \quad \alpha_2, \quad \cdots$$

has a limit λ, A *is regular.*

For, by definition,

$$\epsilon > 0, \quad m, \quad |\lambda - \alpha_n| < \epsilon/2, \quad n > m.$$

$$|\lambda - \alpha_\nu| < \epsilon/2, \quad \nu > m.$$

Adding, by 87, 3,

$$|\alpha_n - \alpha_\nu| < \epsilon. \quad n, \nu > m.$$

Hence A is regular, by 97.

101. 1. *Conversely, if* $A = \alpha_1, \alpha_2, \cdots$ *is a regular sequence, there exists one, and only one, number* α, *such that*

$$\lim \alpha_n = \alpha. \tag*{(1}$$

To show that A cannot have two limits, we need only to repeat the reasoning of 47.

We show now *A has a limit.*

Let
$$\delta_1,\ \delta_2,\ \delta_3,\ \cdots \tag{2}$$
be a sequence of positive numbers whose limit is 0. We choose the δ's now, such that
$$a_n = \alpha_n + \delta_n, \quad n = 1,\ 2,\ \cdots \tag{3}$$
are rational. This is evidently possible by 82. The sequence
$$a_1,\ a_2,\ a_3,\ \cdots \tag{4}$$
is regular.

For,
$$a_n - a_\nu = \alpha_n - \alpha_\nu + (\delta_n - \delta_\nu). \tag{5}$$

Since A is regular,
$$\epsilon > 0,\ m,\ |\alpha_n - \alpha_\nu| < \frac{\epsilon}{2}, \quad n,\ \nu > m. \tag{6}$$

Since, by 100, the sequence 2) is regular,
$$|\delta_n - \delta_\nu| < \epsilon/2. \qquad n,\ \nu > m. \tag{7}$$

In the inequalities 6), 7), we may take m the same. Then 5), 6), 7) give
$$|a_n - a_\nu| \le |\alpha_n - \alpha_\nu| + |\delta_n - \delta_\nu|$$
$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Hence 4) is regular.

We set
$$\alpha = (a_1,\ a_2,\ \cdots).$$
Then, by 97,
$$\lim a_n = \alpha.$$
But, by 3),
$$\alpha_n = a_n - \delta_n.$$
Hence, by 98,
$$\lim \alpha_n = \lim a_n - \lim \delta_n$$
$$= \alpha - 0$$
$$= \alpha.$$

2. As a result of 1 and 100, we have:

In order that a sequence $\alpha_1,\ \alpha_2,\ \cdots$ has a limit, it is necessary and sufficient that it is regular.

102. Let $A = \alpha_1, \alpha_2, \cdots$ be a sequence. Let us pick out of A a sequence

$$B = \alpha_{\iota_1}, \ \alpha_{\iota_2}, \ \cdots$$

where $\iota_1 < \iota_2 < \iota_3, \cdots$. We call B a *partial sequence* of A.

EXAMPLES

1.

$$A = 1, \ \frac{1}{2}, \ \frac{1}{3}, \ \frac{1}{4}, \ \frac{1}{5}, \ \cdots$$

$$B = 1, \ \frac{1}{3}, \ \frac{1}{5}, \ \frac{1}{7}, \ \cdots$$

$$C = 1, \ \frac{1}{2^2}, \ \frac{1}{2^3}, \ \frac{1}{2^4}, \ \cdots$$

$$D = 1, \ \frac{1}{2\cdot 3}, \ \frac{1}{2\cdot 5}, \ \frac{1}{2\cdot 7}, \ \cdots$$

Here B, C, D are partial sequences of A.

2.

$$A = 1, \ \frac{1}{2}, \ 1, \ \frac{1}{3}, \ 1, \ \frac{1}{4}, \ \cdots$$

$$B = 1, \ 1, \ 1, \ \cdots$$

$$C = \frac{1}{2}, \ \frac{1}{3}, \ \frac{1}{4}, \ \cdots$$

B and C are partial sequences of A.

103. 1. Among the symbols given in 42, to indicate the limit of a sequence

$$A = \alpha_1, \ \alpha_2, \ \cdots$$

one was

$$\lim_A \alpha_n.$$

Analogously, we shall denote the limit of a partial sequence

$$B = \alpha_{\iota_1}, \ \alpha_{\iota_2}, \ \cdots$$

of A, by

$$\lim_B \alpha_n.$$

2. We have then, obviously:

If A is regular, so is every partial sequence B; and

$$\lim_A \alpha_n = \lim_B \alpha_n.$$

3. From this, we conclude at once:

The sequence A cannot be regular, if it contains two partial sequences B, C, such that

$$\lim_B \alpha_n \neq \lim_C \alpha_n.$$

4. *The sequence A cannot be regular, if it contains a partial sequence B which is not regular.*

5. It is sometimes a difficult matter to show that a sequence A is or is not regular. The theorems 3, 4 enable us often to show with ease that A is not regular.

Thus, in Ex. 2, 102,

$$\lim_B \alpha_n = 1, \ \lim_C \alpha_n = 0.$$

Hence A is not regular.

6. Unless the contrary is stated, it is to be understood that

$$\lim \alpha_n$$

has reference to the *whole* sequence A.

104. 1. From 98, we conclude the following theorems, which are often useful:

If $\lim (\alpha_n \pm \beta_n) = \sigma$, *and* $\lim \alpha_n = \alpha$; *then* $\lim \beta_n$ *exists and equals* $\pm \sigma \mp \alpha$.

2. *If* $\lim \alpha_n \beta_n = \pi$, *and* $\lim \alpha_n = \alpha \neq 0$; *then* $\lim \beta_n$ *exists and equals* π / α.

3. *If* $\lim \frac{\alpha_n}{\beta_n} = \rho$, *and* $\lim \beta_n = \beta$; *then* $\lim \alpha_n$ *exists and equals* $\beta\rho$.

4. *If* $\lim \frac{\alpha_n}{\beta_n} = \rho \neq 0$, *and* $\lim \alpha_n = \alpha$; *then* $\lim \beta_n$ *exists and equals* α / ρ.

The demonstration of these theorems we illustrate by proving **1.**

We have $$\beta_n = \pm (\alpha_n \pm \beta_n) \mp \alpha_n.$$

Applying 98, 1),

$$\lim \beta_n = \pm \lim (\alpha_n \pm \beta_n) \mp \lim \alpha_n = \pm \sigma \mp \alpha.$$

105. 1. *Let* lim $\alpha_n = \alpha$; *let* β, γ *be two numbers, such that* $\beta < \alpha < \gamma$. *Then*

$$\beta < \alpha_n < \gamma. \qquad n > m.$$

The demonstration is the same as in 48.

2. *Let* lim $\alpha_n = \alpha$, *and* $\alpha_n < \alpha$. *Let* β *be any number* $< \alpha$. *Then*

$$\beta < \alpha_n < \alpha. \qquad n > m.$$

106. 1. *Let* lim $\alpha_n = \alpha$; *if* $\lambda \leq \alpha_n \leq \mu$ *for* $n > m$; *then*

$$\lambda \leq \alpha \leq \mu.$$

For, suppose $\alpha > \mu$. Let β be chosen so that

$$\mu < \beta < \alpha.$$

Then, by 105, 1, $\alpha_n > \beta. \qquad n > m.$

Hence $\alpha_n > \mu,$

which is a contradiction.

2. *If* $\alpha_n \leq \lambda \leq \beta_n,$

and if lim $\alpha_n =$ lim $\beta_n = \mu$;

then $\lambda = \mu.$

For, by 1, $\mu \leq \lambda, \ \mu \geqq \lambda.$

Hence $\mu = \lambda.$

107. *If*

$$\alpha_n \leq \beta_n \leq \gamma_n, \tag{1}$$

and lim $\alpha_n =$ lim $\gamma_n = \lambda$;

then lim $\beta_n = \lambda.$

For, subtracting α_n in 1), we get

$$0 \leq \beta_n - \alpha_n \leq \gamma_n - \alpha_n. \tag{2}$$

Now

$$\lim (\gamma_n - \alpha_n) = \lim \gamma_n - \lim \alpha_n = \lambda - \lambda = 0. \tag{3}$$

But 3) states that $\epsilon > 0, \ m, \ \gamma_n - \alpha_n < \epsilon. \qquad n > m.$

Then, by 2), $\beta_n - \alpha_n < \epsilon.$

This relation states that

$$\lim (\beta_n - \alpha_n) = 0.$$

As $\lim \alpha_n = \lambda$, we have, by 104, 1,

$$\lim \beta_n = \lambda.$$

108. 1. A sequence $A = \alpha_1, \alpha_2, \cdots$, whose elements satisfy the relations

$$\alpha_n < \alpha_{n+1}, \quad n = 1, 2,$$

is called an *increasing sequence.*

If

$$\alpha_n > \alpha_{n+1}, \quad n = 1, 2, \cdots$$

it is a *decreasing sequence.*

If it is either one or the other, but we do not care to specify which, we may call it a *univariant sequence.*

If, on the other hand,

$$\alpha_n \leq \alpha_{n+1}, \quad n = 1, 2, \cdots$$

A is said to be a *monotone increasing sequence.*

If

$$\alpha_n \geqq \alpha_{n+1}, \quad n = 1, 2, \cdots$$

it is a *monotone decreasing sequence.*

If A is either one or the other, but we do not care to specify which, we may call A a *monotone sequence.*

Univariant sequences are special cases of monotone sequences

2. If there exists a fixed positive number G, such that

$$|\alpha_n| \leq G, \quad n = 1, 2, \cdots$$

A is said to be *limited*, otherwise *unlimited.*

109. *A limited monotone sequence is regular.*

For clearness, let $A = \alpha_1, \alpha_2, \cdots$ be an increasing monotone sequence, and let

$$\alpha_n < G. \qquad n = 1, 2, \cdots \tag{1}$$

To show that A is regular, we must show that

$$\epsilon > 0, \ m, \ 0 \leq \alpha_n - \alpha_m < \epsilon, \qquad n > m. \tag{2}$$

Since A is monotone increasing,

$$0 \leq \alpha_n - \alpha_m.$$

To show the rest of 2), take m_0 at pleasure. Either there exists an infinite sequence of indices

$$m_0 < m_1 < m_2 < \cdots \tag{3}$$

such that

$$\alpha_{m_1} - \alpha_{m_0} \geqq \epsilon, \quad \alpha_{m_2} - \alpha_{m_1} \geqq \epsilon, \quad \cdots \tag{4}$$

or there does not.

Suppose such a sequence 3) exists. Then, however small ϵ has been taken, we can take the integer p so large that

$$p\epsilon + \alpha_{m_0} > G. \tag{5}$$

Adding the first p inequalities 4), we get

$$\alpha_{m_p} - \alpha_{m_0} \geqq p\epsilon.$$

Hence, by 5),

$$\alpha_{m_p} > G,$$

which contradicts 1).

We thus conclude that there exist but a finite number of indices m_t, such that 4) holds. Thus we can take m so large that

$$\alpha_n - \alpha_m < \epsilon, \qquad n > m,$$

which proves the other half of 2).

110. 1. A limited increasing sequence of great importance is

$$a_n = \left(1 + \frac{1}{n}\right)^n. \qquad n = 1, 2, \cdots \tag{1}$$

To show that 1) is increasing, *i.e.* that

$$a_n > a_{n-1}, \tag{2}$$

we employ the relation 94, 5), viz.:

$$\beta^n > \alpha^{n-1}[\alpha - n(\alpha - \beta)]. \tag{3}$$

Set

$$\alpha = 1 + \frac{1}{n-1}, \quad \beta = 1 + \frac{1}{n},$$

in 3); we get 2) at once, for $n \geqq 2$.

To show that 1) is limited, we set

$$\alpha = 1 + \frac{1}{2m}, \quad \beta = 1, \qquad n = m + 1,$$

in 3); we get

$$1 > \frac{1}{2}\left(1 + \frac{1}{2m}\right)^m; \qquad m = 1, 2, \cdots$$

or squaring,

$$4 > \left(1 + \frac{1}{2m}\right)^{2m}.$$

Thus

$$a_{2m} < 4. \tag*{(4}$$

But, by 2),

$$a_{2m-1} < a_{2m}. \tag*{(5}$$

As all positive integers are of the form

$$2m \text{ or } 2m - 1,$$

4) and 5) give

$$a_n < 4. \qquad n = 1, 2, \cdots$$

2. Since the sequence 1) is limited and monotone, it has a limit by 109. We set

$$e = \lim_{n=\infty}\left(1 + \frac{1}{n}\right)^n. \qquad n = 1, 2, 3, \cdots$$

As the reader already knows, $e = 2.71828 \cdots$, and is the base of the Napierian system of logarithms.

111. 1. *Let* $A = \alpha_1, \alpha_2, \cdots$ *be a regular sequence, whose limit is* α. *In* A *exist partial monotone sequences* B; *and for each such sequence,*

$$\lim_B \alpha_n = \alpha.$$

Then are two cases: 1° $\alpha - \alpha_n$, $n > m$ has one sign, when not zero; 2° $\alpha - \alpha_n$ may have both signs, however large m is taken.

Case I. To fix the ideas, suppose

$$\alpha - \alpha_n \geqq 0. \qquad n \geq m. \tag*{(1}$$

In this relation, it may happen that for some $m' \geq m$

$$\alpha - \alpha_n = 0. \qquad n \geqq m'.$$

In this case,

$$B = \alpha_m,\ \alpha_{m+1}$$

$$= \alpha,\ \alpha,\ \cdots$$

is a sequence required in the theorem.

Let us suppose now that there are in 1) an infinite number of indices n_κ, such that

$$\alpha - \alpha_{n_\kappa} > 0.$$

Let ν_1 be one of the indices n_κ; then

$$\alpha_{\nu_1} < \alpha.$$

Let β_1 lie between these, so that

$$\alpha_{\nu_1} < \beta_1 < \alpha.$$

Then, by 105, 1, and 103, 2, there are an infinity of elements α_{n_κ}, lying between β_1 and α. Let α_{ν_2} be one of these, so that

$$\beta_1 < \alpha_{\nu_2} < \alpha.$$

Let β_2 lie between α_{ν_2} and α; then for some index ν_3 we have

$$\beta_2 < \alpha_{\nu_3} < \alpha.$$

In this way we find an *increasing sequence*

$$B = \alpha_{\nu_1},\ \alpha_{\nu_2},\ \alpha_{\nu_3},\ \cdots$$

which is a partial sequence.

Then, by 103, 2,

$$\lim_{n=\infty} \alpha_{\nu_n} = \alpha.$$

The number of sequences B of this type is obviously unlimited.

Case II. Since there are an unlimited number of indices for which

$$\alpha - \alpha_n > 0, \tag{2}$$

let us denote those values of n for which 2) holds, by $n_1, n_2, n_3, \cdots$.

Then the partial sequence of A,

$$A' = \alpha_{n_1},\ \alpha_{n_2},\ \cdots$$

belongs to Case I. Hence in A' lie an infinity of sequences of the type B.

2. The demonstration of 1 shows:

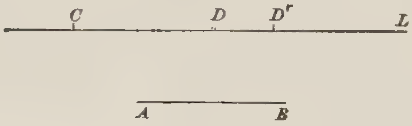
If, in the regular sequence $A = \alpha_1, \alpha_2, \cdots$, *the* α_n *do not finally become all equal, there exists in* A *an infinity of partial univariant sequences* B *which have all the same limit as* A.

The Measurement of Rectilinear Segments. Distance

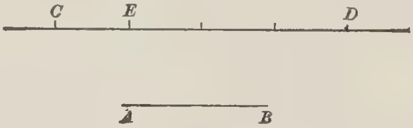
112. In 39, 43, 44, we have made use of the graphical representation of the numbers in R, by points of a right line. We wish now to extend the considerations to numbers in $\mathfrak{R}$. With this end in view, we proceed to develop the theory of *measurement of rectilinear segments* and the associated notion of *distance*.

113. 1. Let AB, CD be two rectilinear segments. We say AB is *greater* than CD, when, if superimposed, AB contains CD as a part; while CD is said to be *less* than AB. If, when superimposed, AB and CD coincide, we say AB and CD are *equal*.

2. We assume, with *Archimedes*, that if the segment AB is laid off a sufficient number of times on the line L, we can obtain a



segment CD' greater than any given segment CD. And conversely, that it is possible to divide a segment CD into a sufficient number



of equal parts, so that one of them, as CE, is less than any given segment AB.

114. 1. Let $S = AB$ be a segment we wish to measure; and let $U = CD$ be a segment which we take as a unit of comparison.

If we can divide S into l equal segments, equal to U, *i.e.* if

$$S = l \cdot U,$$

we say l is the measure of S, or l is the length of S.

2. If it is impossible to do this, it may happen that n segments S are equal to m segments U; *i.e.*

$$n \cdot S = m \cdot U.$$

We say then, that

$$l = \frac{m}{n}$$

is the measure or length of S.

3. In both cases we say S is *commensurable* with U.

The segment AB being commensurable, we say its length l expresses *the distance of A from B*, or B from A. We write

$$l = \text{Dist}\,(A, B),$$

or more shortly

$$l = \overline{AB}.$$

115. We show now that the number l, just determined, is unique. This is evident when l is an integer. We suppose, therefore, that

$$nS = mU, \tag{1}$$

and

$$n_1 S = m_1 U. \tag{2}$$

Multiplying these equations respectively by n_1, n, we get

$$nn_1 S = n_1 m U, \quad nn_1 S = nm_1 U.$$

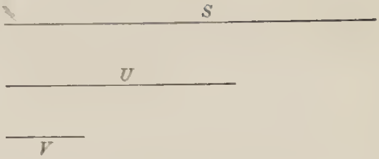
$$\therefore\ n_1 m U = nm_1 U.$$

$$\therefore\ n_1 m = nm_1,$$

or

$$\frac{m}{n} = \frac{m'}{n'}.$$

Thus, the two equations 1), 2) lead to the same value of l.



116. 1. Let

$$l = \frac{m}{n}$$

be the measure of S.

Then $$nS = mU. \qquad (1$$

Let us divide U into n equal parts, and call V one of them. Then

$$nV = U.$$

This in 1) gives $$nS = mnV.$$

Hence $$S = mV.$$

This shows that by taking a *new unit* V, whose length is $1/n$ of the old unit, the length of S can be expressed as an integer.

2. The above considerations also give us a new way for defining the length of S. In fact, suppose it possible to divide U into s equal parts V, such that

$$S = rV. \qquad (2$$

Then

$$l = \frac{r}{s}. \qquad (3$$

For, $$sV = U;$$

hence, multiplying 2) by s, we get

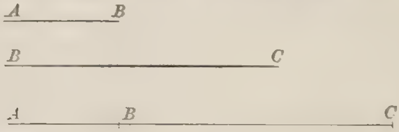
$$sS = rsV = rU;$$

so that the length l of S is indeed given by 3).

117. Let $S = AB$, $T = BC$ be two segments whose lengths are respectively

$$l = \frac{a}{b}, \quad m = \frac{c}{d};$$

a, b, c, d being positive integers



If we put them end to end, we get a segment $W = AC$ whose length, we show, is

$$n = l + m.$$

By definition we have

$$bS = aU, \quad dT = cU.$$

Multiplying these equations respectively by d, b, and adding, we get

$$bd \cdot S + bd \cdot T = adU + bcU = (ad + bc)U. \tag{1}$$

But

$$bd \cdot S + bd \cdot T = bd \cdot W. \tag{2}$$

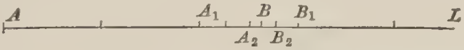
Hence 1), 2) give

$$bdW = (ad + bc)U.$$

Hence

$$n = \frac{ad + bc}{bd} = \frac{a}{b} + \frac{c}{d} = l + m.$$

118. 1. We turn now to the measurement of segments which are *incommensurable* with the segment chosen as unit. An example of such segments is the diagonal of a square, the side being taken as unit.



To measure AB, we begin by marking off points on the right line L, at a unit distance apart, starting with A. By the axiom of Archimedes, 113, 2, B will fall between two consecutive points of this set, say between A_1, B_1.

Let

$$l_1 = \text{Dist}(A, A_1).$$

On the segment A_1B_1 we mark off points at the distance $1/n$ apart, where n is an arbitrary positive integer.

Then B will fall between two of these points which are consecutive, say between A_2, B_2.

Let

$$l_2 = \text{Dist}(A, A_2).$$

We may continue in this way, subdividing each interval A_m, B_m into n equal parts, without end. The point B will never fall on the end point of one of these intervals, for then AB would be commensurable. The sequence of rational numbers

$$l_1, \ l_2, \ l_3, \ \cdots \tag{1}$$

so determined is monotone increasing, and limited. In fact, all its elements are $< l_1 + 1$. Thus, by 109, the sequence 1) is regular, and so defines a number λ. We say λ is the *measure* or *length* of AB, and we write as before

$$\lambda = \text{Dist}\,(A,\ B) = \overline{A,\ B}.$$

2. If we had taken the numbers

$$l_\kappa' = \text{Dist}\,(A,\ B_\kappa), \qquad \kappa = 1,\ 2,\ \cdots$$

where B_κ denotes the right-hand end of the interval in which B falls, instead of the numbers l_κ, we would have got a monotone decreasing limited sequence

$$l_1',\ l_2',\ l_3',\ \cdots$$

whose limit $\lambda' = \lambda$.

For,

$$l_\kappa' - l_\kappa = \frac{1}{n^{\kappa-1}},$$

whose limit is 0.

119. We have defined the length λ of AB by a process which subdivides each interval A_κ, B_κ into n equal parts. The question at once arises: would this process lead to the same number λ, if we had divided each interval into m instead of n equal parts?

We prove the following general result: Let us modify the above process so as to divide the first interval into n_1 equal parts, the second interval into n_2 equal parts, etc. This system of subdivision leads to a sequence which we denote by

$$l_1',\ l_2',\ l_3',\ \cdots \tag{1}$$

where

$$l_m' = \text{Dist}\,(A,\ A_m').$$

The limit of 1) being λ', we show it exists and $\lambda = \lambda'$.

For, each point A_m' will fall in a certain interval A_{ι_m}, B_{ι_m} of the old system of subdivision, where ι_m is the lowest index for which this is true.

Then

$$\text{Dist}\,(AA_{\iota_m}) \gtreqless \text{Dist}\,(A,\ A_m') \leq \text{Dist}\,(AB_{\iota_m}).$$

But, by 118, 2,

$$\lim \text{Dist}\,(A,\ A_{\iota_m}) = \lim \text{Dist}\,(A,\ B_{\iota_m}) = \lambda.$$

Hence, by 107,

$$\lim \text{Dist}\,(A,\ A_m') = \lambda.$$

120. The process explained in 118, 119 is obviously applicable to the case when AB is commensurable. The only difference is that after a certain number of steps the point B may fall on one of the end points of the little segments A_m, B_m. In this case the corresponding sequence

$$l_1,\ l_2,\ \cdots\ l_s,\ l_s,\ l_s,\ \cdots$$

would have all its elements the same after a certain one.

121. We have now two methods for measuring a *commensurable* segment; viz. those given in 114 and 120.

Let l be the length of AB as given by 114; and λ its length according to 120. We show

$$l = \lambda.$$

Since AB is, by hypothesis, commensurable, we have, by 117, preserving the notation already employed,

$$\begin{aligned} l &= \text{Dist}\,(AA_m) + \text{Dist}\,(A_mB) \\ &< \text{Dist}\,(AA_m) + \text{Dist}\,(A_mB_m). \qquad (1 \end{aligned}$$

As

$$\text{Dist}\,(A_mB_m) = \frac{1}{n^{m-1}},$$

we have, from 1),

$$0 < l - l_m < \frac{1}{n^{m-1}},$$

where

$$l_m = \text{Dist}\,(A,\ A_m).$$

Passing to the limit, we have, since

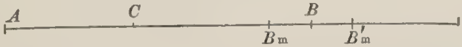
$$\lim l_m = \lambda,\ \lim \frac{1}{n^{m-1}} = 0,$$

$$l = \lambda.$$

122. 1. Let AC, CB be any two segments; we show that

$$\text{Dist}\,(AB) = \text{Dist}\,(AC) + \text{Dist}\,(CB).$$

This is a generalization of 117.



We begin by supposing that one of the segments, AC, is commensurable, while the other, CB, is not. Then AB is not commensurable. In our process of subdivision, suppose that after the mth step, B falls in the segment B_m, B_m'. Then AC and CB_m are commensurable. Hence, by 117,

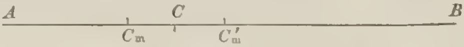
$$\text{Dist}\,(AB_m) = \text{Dist}\,(AC) + \text{Dist}\,(CB_m).$$

In the limit, we get

$$\text{Dist}\,(AB) = \text{Dist}\,(AC) + \text{Dist}\,(CB).$$

2. We pass now to the general case.

After the mth subdivision,



let C fall between C_m and C_m'. Then AC_m is commensurable. Hence, by 1,

$$\text{Dist}\,(AB) = \text{Dist}\,(AC_m) + \text{Dist}\,(C_mB).$$

Passing to the limit, we have

$$\text{Dist}\,(AB) = \text{Dist}\,(AC) + \text{Dist}\,(CB).$$

Correspondence between $\mathfrak{R}$ and the Points of a Right Line

123. 1. On the indefinite right line L mark a point O as origin. Let P be any point on L, and let

$$\lambda = \text{Dist}\,(O, P).$$

If P lies to the right of O, we associate with P the number $+\lambda$; if P lies to the left of O, we associate with it $-\lambda$. With the origin we associate the number 0. Thus to any point on L corresponds a number in $\mathfrak{R}$, and to different points correspond different numbers.

2. We ask now conversely: does there exist for each λ in $\mathfrak{R}$, a point P such that

$$|\lambda| = \text{Dist}\,(O, P)\,?$$

For all rational numbers this is true by virtue of the axiom of Archimedes, 113, 2.

Whether it is true for every number in $\mathfrak{R}$, cannot be demonstrated. We therefore *assume* with Dedekind and Cantor that there shall exist one and only one point P which shall lie to the right of O, if $\lambda > 0$, and to the left of O, if $\lambda < 0$, and such that

$$|\lambda| = \text{Dist}\,(O,\ P).$$

This we shall call the *Cantor-Dedekind axiom* or the *axiom of continuity of the right line.* As we proceed, it will be made evident that many apparently simple geometric ideas are extremely subtle and complex. One of the most elusive of these is the notion of continuity. To say the right line is continuous because it has no *breaks* or *gaps*, is simply to replace one undefined word by another.

3. We have now established a *one to one correspondence* between the numbers of $\mathfrak{R}$ and the points on L. We may consider the points as *images* or *representations* of these numbers.

124. 1. The correspondence which we have just defined is a generalization of that given in 35 for R. The considerations of 39, 43, 44 can now be extended to $\mathfrak{R}$ without any further comment. The graphical interpretation of sequences and their limits which we thus obtain will illuminate greatly the section on *limits*, 97–111. We recommend the student to go over the demonstrations which we gave there, employing graphical representations as an aid to the reasoning. Indeed, some of the theorems, when interpreted geometrically, seem almost self-evident.

2. Consider, for example, the theorem of 107.

We have there three sequences, $A = \{\alpha_n\}$, $B = \{\beta_n\}$, $C = \{\gamma_n\}$.

The relation

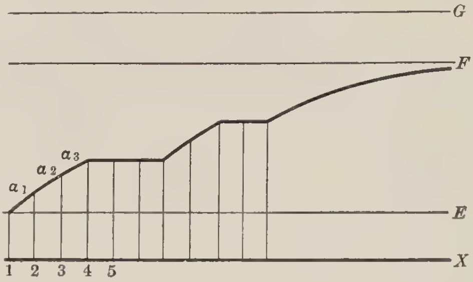
$$\alpha_n \leq \beta_n \leq \gamma_n$$

states that the point β_n lies in the interval $I_n = (\alpha_n,\ \gamma_n)$.

Since now both end points of I_n converge to the point λ, evidently *any* point in I_n, as β_n, must also converge to the point λ.

125. As another example, consider the theorem of 109.

To fix the ideas, let $A = \{\alpha_n\}$ be a monotone increasing sequence. The points in the figure represent $\alpha_1,\ \alpha_2,\ \cdots$ We have drawn a curve through them, as the eye seizes more easily the law of increase or decrease of a sequence when such a curve is drawn. The reader will observe that since the sequence is monotone, this curve can have segments parallel to the axis X. As A is limited, all the points of A lie between a certain line G, and a line E drawn through the first point α_1 of the sequence. To see now that A must have a limit, let us suppose the line G moved parallel to itself toward X. Evidently there is a line F below which G cannot move without getting below points of A, and which the points of A approach as an asymptote.



If λ is the distance of F from X, evidently

$$\lim \alpha_n = \lambda.$$

126. As a final example of the helpfulness of graphical representation, let us consider the theorem of 111.

The two cases we considered there are represented in Figs. 1 and 2. The heavy curves represent the law of increase and decrease of the sequence A. The points α_1, α_2, $\cdots$ lie on these curves, but are not indicated. The straight lines A represent the limit α of A. The light curve in Fig. 1 indicates an increasing sequence which one could pick out of A.

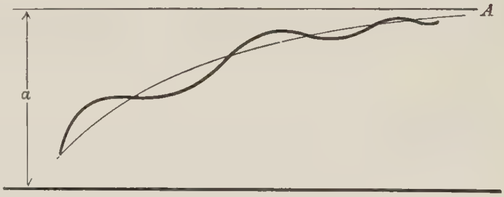


FIG. 1.

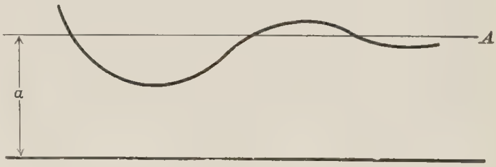


FIG. 2.

By the aid of such a representation the theorem becomes almost self-evident.

127. 1. *Let* $A = \{\alpha_n\}$, $B = \{\beta_n\}$, $C = \{\gamma_n\}$.

Let A be monotone increasing, and C monotone decreasing. Let

$$\alpha_n \leq \beta_n \leq \gamma_n, \tag{1}$$

and

$$\lim (\gamma_n - \alpha_n) = 0. \tag{2}$$

Then A, B, C are regular, and have the same limit λ. *Also*

$$\alpha_n \leq \lambda \leq \gamma_n. \tag{3}$$

Graphically, the theorem is obviously true.

The points α_n, γ_n determine a set of intervals

$$I_n = (\alpha_n, \gamma_n),$$

such that each I_n lies in the preceding I_{n-1}, and hence in all preceding intervals.

By 2) the lengths of these intervals converge to 0. Geometrically, it is evident that the end points α_n, γ_n of these intervals converge to the same limiting point λ, and that any sequence of points β_n, where β_n is any point in I_n, must also converge to λ.

Arithmetically, the demonstration is as follows:

By hypothesis,

$$\alpha_1 \leq \alpha_2 \leq \alpha_3 \cdots$$

$$\gamma_1 \geq \gamma_2 \geq \gamma_3 \cdots \tag{4}$$

From 1),

$$\alpha_n \leq \gamma_n.$$

Hence, by 4),

$$\alpha_n \leq \gamma_1. \qquad n = 1, 2, \cdots$$

Thus A is limited. Similarly, C is limited.

Thus A and C are regular, by 109.

Let

$$\lim \alpha_n = \lambda. \tag{5}$$

Then 2) and 5) give, by 104, 1,

$$\lim \gamma_n = \lambda.$$

Then, by 107,

$$\lim \beta_n = \lambda.$$

2. The preceding theorem may be put in geometrical language, and gives:

Let $I_n = (\alpha_n, \gamma_n)$ *be a sequence of intervals* $n = 1, 2, 3, \cdots$. *Let* I_n *lie in* I_{n-1}, *and let the lengths of these intervals converge to* 0. *Let* β_n *be any point in* I_n *(including end points). Then the sequence* $\{\beta_n\}$ *is regular, and all such sequences have the same limit* λ. *The point* λ *lies in every* I.

3. The reader should avoid the following *error:*

Let $\{\alpha_n\}$, $\{\beta_n\}$ be two sequences, such that

$$\lim (\alpha_n - \beta_n) = 0. \qquad (6$$

Then $\lim \alpha_n$, $\lim \beta_n$ exist and are equal.

That this conclusion is false is shown by the following example:

$$\alpha_n = (-1)^n + \frac{1}{n}, \quad \beta_n = (-1)^n.$$

Here neither limit exists, although 6) is satisfied.

Dedekind's Partitions

128. We proceed now to establish the notion of *partition*,* introduced by Dedekind, to found his theory of irrational numbers.

Let α be any number of $\mathfrak{R}$; we can use it to throw all numbers of $\mathfrak{R}$ into two classes A, B. In A we put all numbers $< \alpha$; in B all numbers $> \alpha$. The number α we may put in A or B. This division of the numbers of $\mathfrak{R}$ into two classes we call a *partition*, and say, α *generates a partition* (A, B). Geometrically, the point α may be used to throw all points of a right line into two classes. In class A we put all points to the left of α; in B all points to the right of α. The point α we put in either A or B at pleasure.

Example.

Let $\alpha = \sqrt{2}$. In A put all numbers $< \sqrt{2}$; in B put the numbers $\geqq \sqrt{2}$.

This partition may also be generated as follows: in A put all numbers whose square is < 2; in B all numbers whose square $\geqq 2$.

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* The German term is *Schnitt*.

129. 1. *More generally*, we shall say that any separation of the numbers of $\mathfrak{R}$ into two classes A, B, such that

1° Any number of A is $<$ any number in B,

2° Any number of B is $>$ any number in A,

constitutes a partition (A, B).

2. The condition 2° is really redundant, as it follows at once from 1°.

In fact, suppose 2° did not follow from 1°; *i.e.* suppose there were a number β in B, $\overline{\overline{<}}$ some number α in A. Then there is an α in A which is not $<$ any number in B, for α is not $<\beta$. This is a contradiction.

3. Two partitions (A, B) and (C, D) are the *same* or *equal* only when A and C contain the same set of numbers; or only when B and D contain the same numbers, excepting possibly the end numbers.

130. 1. We consider now this question: suppose a law given which throws all numbers into two classes A, B, such that every number in A is less than any number in B, and every number in B is greater than any number in A. Is there a number λ in $\mathfrak{R}$ which will generate this partition (A, B)? We show there is.

To this end we construct two sequences

$$S = \alpha_1,\ \alpha_2,\ \alpha_3,\ \cdots$$

$$T = \beta_1,\ \beta_2,\ \beta_3,\ \cdots$$

S being monotone increasing, and T monotone decreasing, as follows:

Let α_1 be any number at pleasure in A, and β_1 a number in B. Their arithmetic mean

$$\frac{\alpha_1 + \beta_1}{2}$$

lies between α_1, β_1.

If it lies in A, we set

$$\alpha_2 = \tfrac{1}{2}(\alpha_1 + \beta_1),\ \beta_2 = \beta_1;$$

if it lies in B, we set

$$\alpha_2 = \alpha_1,\ \beta_2 = \tfrac{1}{2}(\alpha_1 + \beta_1).$$

We build now the arithmetic mean of α_2, β_2, and reason with this as before. Continuing this process indefinitely, we get the sequences S and T.

By 127, 2, the sequences S, T have a common limit, which we call λ.

Let λ generate the partition (A', B').

We show that

$$(A, B) = (A', B'),$$

by showing 1° that every number in A' lies in A; and 2° that every number of A lies in A'.

1°. Let $\alpha' \neq \lambda$ be any number of A'. By 105, 2 there are an infinity of numbers α_n lying between α' and λ. But α_n is in A, by hypothesis. Hence $\alpha' < \alpha_n$ is in A.

2°. Let α be any number of A. We have to show that $\alpha \overline{\leq} \lambda$. Suppose the contrary, $\lambda < \alpha$.

Then

$$\epsilon = \alpha - \lambda > 0.$$

We can take n so great that

$$\beta_n - \alpha_n = \frac{\beta_1 - \alpha_1}{2^{n-1}} < \epsilon. \tag{1}$$

On the other hand, by supposition,

$$\alpha_n < \lambda < \alpha < \beta_n.$$

Hence

$$\beta_n - \alpha_n \geq \alpha - \lambda = \epsilon,$$

which contradicts 1).

2. We have thus this theorem:

Every partition can be generated by a number in $\mathfrak{R}$.

131. 1. *A partition* (A, B) *cannot be generated by two different numbers* λ and μ.

To fix the ideas, let $\lambda < \mu$.

In the partition (C, D) generated by μ, C contains all numbers $< \mu$. It therefore contains numbers $> \lambda$, and hence numbers not in A. Hence (A, B), (C, D) are different.

2. Since each number generates *one* partition, and each partition is generated by *one* number, we can establish a uniform or one to

one correspondence between the numbers of $\mathfrak{R}$ and the aggregate of all possible partitions.

In fact, to the number α shall correspond the partition (A, B), that α generates. To the partition (F, G) shall correspond the number λ, which will generate (F, G).

Infinite Limits

132. Let $A = \alpha_1, \alpha_2, \cdots$ be an *unlimited* sequence [108, 2]. The following cases may occur:

1°. For each positive number G, arbitrarily large, there exists an m, such that $\alpha_n > G$, for every $n \geq m$. In symbols

$$G > 0, \quad m, \quad \alpha_n > G, \quad n \geq m.$$

We say the limit of A is *plus infinity;* and write

$$\lim_{n=\infty} \alpha_n = +\infty, \quad \lim \alpha_n = +\infty, \quad \alpha_n \doteq +\infty.$$

Such sequences are

$$1,\ 2,\ 3,\ \cdots$$

$$1!,\ 2!,\ 3!,\ \cdots$$

2°. For each negative number G, arbitrarily large, there exists an m, such that $\alpha_n < G$, for every $n \geqq m$. In symbols

$$G < 0, \quad m, \quad \alpha_n < G, \quad n \geqq m.$$

We say the limit of A is *minus infinity;* and write

$$\lim_{n=\infty} \alpha_n = -\infty, \quad \lim \alpha_n = -\infty, \quad \alpha_n$$

Such a sequence is

$$-10, \quad -100, \quad -1000, \quad \cdots$$

In both these cases, we say the limit is *definitely or determinately infinite.*

3°. The elements α_n do not finally all have one sign, but still

$$\lim |\alpha_n| = +\infty.$$

We say the limit of A is *indefinitely or indeterminately infinite.* Such a sequence is

$$1. \quad -2, \quad +3, \quad -4, \quad +5, \quad \cdots$$

133. 1. Case 3 is of little importance. We shall therefore in the future, when using the terms the *limit is infinite*, or certain variables *become infinite*, always mean *definitely infinite*, unless the contrary is expressly stated.

The symbol $+\infty$ is frequently written without the + sign.

The symbol $\pm\infty$ means that the limit is either $+\infty$ or $-\infty$, and one does not care to specify which.

The limits defined in the previous sections are called, in contradistinction, *finite limits*.

The symbols $+\infty$, $-\infty$ are not numbers; *i.e.* they do not lie in $\mathfrak{R}$. They are introduced to express shortly certain *modes of variation* which occur constantly in our reasonings.

2. Finally, we wish to state, *once for all*, that the terms, the *limit exists*, the *limit is* λ, etc., or an *equation* as

$$\lim \alpha_n = \lambda,$$

always refer to finite limits, unless the supplementary phrase, "finite or infinite," is inserted.

134. *A sequence cannot have both a finite and an infinite limit.*

For, if $A = \{\alpha_n\}$ has a finite limit, the numbers α_n lie between two fixed numbers, by 105, 1. It is thus limited. It cannot therefore have an infinite limit.

135. *Let* $A = \{\alpha_n\}$, *and let* B *be any partial sequence of* A. *If*

$$\lim_A \alpha_n = \pm\infty,$$

then

$$\lim_B \alpha_n = \pm\infty.$$

The demonstration is obvious.

136. *If the limit of a sequence* $A = \{\alpha_n\}$ *is indefinitely infinite, its positive and negative terms each form sequences whose limits are respectively* $+\infty$ *and* $-\infty$.

For, let $B = \{\beta_n\}$ be the sequence formed of the positive terms of A; and $C = \{\gamma_n\}$ the sequence formed of the negative terms.

By hypothesis,

$$G > 0, \quad m, \quad |\alpha_n| > G, \quad n \gtreqless m.$$

But then, for a sufficiently large m',

$$\beta_n > G, \ \ \gamma_n < -G, \ \ n \geq m'.$$

Hence

$$\lim \beta_n = +\infty, \ \ \lim \gamma_n = -\infty.$$

137. *Let*
$$\lim \alpha_n = \alpha, \ \ \lim \beta_n = \pm\infty;$$

then 1.
$$\lim (\alpha_n \pm \beta_n) = \pm\infty, \ \ \lim \frac{\alpha_n}{\beta_n} = 0.$$

2. *If* $\alpha \neq 0$,
$$\lim \alpha_n \beta_n = \pm\infty.$$

3. *If* $\alpha \neq 0$,
$$\frac{\alpha_n}{\beta_n} > 0, \ \ \textit{and} \ \ \lim \beta_n = 0;$$

then
$$\lim \frac{\alpha_n}{\beta_n} = +\infty.$$

The demonstration is obvious.

138. *Let*
$$\alpha_1, \ \alpha_2, \ \alpha_3, \ \cdots \qquad \beta_1, \ \beta_2, \ \beta_3, \ \cdots$$
be two sequences.

Let
$$\beta_n \geqq \alpha_n. \qquad n = 1, 2, \cdots$$

If $\lim \alpha_n = +\infty$,

then
$$\lim \beta_n = +\infty.$$

For, by hypothesis,

$$G > 0, \ \ m, \ \ \alpha_n > G, \ \ n > m.$$

Since
$$\beta_n \geqq \alpha_n,$$
we have, *a fortiori*,
$$\beta_n > G.$$

Hence
$$\lim \beta_n = +\infty.$$

139. *Let* $A = \alpha_1, \alpha_2, \cdots$ *be a monotone increasing sequence. Let*

$$B = \alpha_{\iota_1}, \ \alpha_{\iota_2}, \ \alpha_{\iota_3}, \ \cdots \qquad \iota_1 < \iota_2 < \iota_3 \cdots$$

be a partial sequence of A.

If $\lim_{B} \alpha_n = +\infty$, *then* $\lim_{A} \alpha_n = +\infty$.

For, by hypothesis,

$$G > 0, \quad m, \quad \alpha_{\iota_n} > G. \qquad n > m.$$

But since A is monotone increasing,

$$\alpha_r > \alpha_{\iota_m}. \qquad r > \iota_m.$$

Hence

$$\alpha_r > G,$$

and

$$\lim \alpha_r = +\infty.$$

140. *Let $m_1, m_2, \cdots$ be a sequence of integers whose limit is $+\infty$. Then*

$$\lim \alpha^{m_n} = 0, \text{ if } 0 < \alpha < 1. \tag{1}$$

$$= 1, \text{ if } \alpha = 1. \tag{2}$$

$$= +\infty, \text{ if } \alpha > 1. \tag{3}$$

For, *let* $\alpha > 1$.

We set

$$\alpha = 1 + \delta. \qquad \delta > 0.$$

Then, by 94, 3),

$$\alpha_\kappa = (1 + \delta)^{m_\kappa} > \beta_\kappa, \tag{4}$$

where

$$\beta_\kappa = 1 + m_\kappa \delta.$$

We apply now 138.

Since

$$\lim \beta_\kappa = +\infty,$$

we have, from 4),

$$\lim \alpha_\kappa = +\infty.$$

Let $0 < \alpha < 1$.

We set

$$\rho = \frac{1}{\alpha}.$$

Then

$$\rho > 1.$$

Also

$$\alpha^{m_n} = \frac{1}{\rho^{m_n}}.$$

As by 3),

$$\lim \rho^{m_n} = +\infty,$$

we have, by 137,

$$\lim \alpha^{m_n} = 0,$$

which proves 1).

The truth of 2) is obvious.

141. We consider now a few examples involving *infinite limits*.

Example 1.

$$\alpha_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}. \qquad n = 1, 2, \cdots$$

Here

$$\lim \alpha_n = +\infty. \tag{1}$$

For, let

$$\mu = 2^m - 1. \qquad m = 1, 2, \cdots$$

Then

$$\alpha_\mu = 1 + \left(\frac{1}{2} + \frac{1}{3}\right) + \left(\frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7}\right) + \left(\frac{1}{8} + \frac{1}{9} + \cdots + \frac{1}{15}\right)$$
$$+ \cdots + \left(\frac{1}{2^{m-1}} + \cdots + \frac{1}{2^m - 1}\right).$$

Each parenthesis is $> \frac{1}{2}$.

For,

$$\frac{1}{2} + \frac{1}{3} > \frac{1}{4} + \frac{1}{4} = \frac{1}{2},$$

$$\frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} > \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{1}{2}, \text{ etc.}$$

Thus

$$\alpha_\mu > \frac{m}{2}.$$

As

$$\lim m = +\infty,$$

we have, by 138,

$$\lim \alpha_\mu = +\infty.$$

But $\{\alpha_\mu\}$ is a partial sequence of the increasing sequence $\{\alpha_n\}$. Hence, by 139, we have 1).

142. *Example* 2.

$$\alpha_n = \frac{1}{\alpha} + \frac{1}{\alpha + 1} + \frac{1}{\alpha + 2} + \cdots + \frac{1}{\alpha + n},$$

where $\alpha \neq 0, -1, -2, -3, \cdots$

Then

$$\lim \alpha_n = +\infty. \tag{1}$$

Let $\alpha > 0$.

Then there is a positive integer m, such that

$$m - 1 \leq \alpha < m.$$

Then
$$\frac{1}{\alpha+p} > \frac{1}{m+p}. \qquad p = 0, 1, 2, \cdots$$

Hence
$$\alpha_n > \frac{1}{m} + \frac{1}{m+1} + \cdots + \frac{1}{m+n} = \beta_n.$$

But, by 135, 141,
$$\lim \beta_n = +\infty.$$

Hence, by 138,
$$\lim \alpha_n = +\infty.$$

Let $\alpha < 0$.

Then there exists a positive integer m, such that
$$0 < \alpha + m < 1.$$

Hence
$$\gamma_n = \frac{1}{\alpha+m} + \frac{1}{\alpha+m+1} + \cdots + \frac{1}{\alpha+m+n} > 1 + \frac{1}{2} + \cdots + \frac{1}{n}.$$

Then, by 141 and 138,
$$\lim \gamma_n = +\infty.$$

But $\{\gamma_n\}$ is a partial sequence of $\{\alpha_n\}$. Hence we have again 1), by 139.

143. *Example* 3.
$$Q_n = \frac{\alpha(\alpha+1)\cdots(\alpha+n)}{\beta(\beta+1)\cdots(\beta+n)},$$
where $\beta \neq 0, -1, -2, \cdots$

1°. $\alpha > \beta$, $\beta > 0$. Let $\delta = \alpha - \beta$.

Then
$$\frac{\alpha+m}{\beta+m} = 1 + \frac{\delta}{\beta+m}. \qquad m = 0, 1, \cdots n.$$

Hence
$$Q_n = \left(1+\frac{\delta}{\beta}\right)\left(1+\frac{\delta}{\beta+1}\right)\cdots\left(1+\frac{\delta}{\beta+n}\right)$$
$$> 1 + \delta\left(\frac{1}{\beta} + \frac{1}{\beta+1} + \cdots + \frac{1}{\beta+n}\right),$$
by 90, 1. Hence, by 142,
$$\lim Q_n = +\infty.$$

2°. $\alpha > \beta$, $\beta < 0$.

If α is a negative integer or 0, Q_n finally becomes and remains 0.

Hence
$$\lim Q_n = 0. \qquad \alpha = 0,\ -1,\ -2,\ \cdots$$

Otherwise, let the positive integer m be taken so large that
$$\beta + m > 0.$$

Then
$$Q_n = \frac{\alpha(\alpha+1)\cdots(\alpha+m-1)}{\beta(\beta+1)\cdots(\beta+m-1)} \cdot \frac{(\alpha+m)\cdots(\alpha+n)}{(\beta+m)\cdots(\beta+n)} = RS_n.$$

The first factor R is a constant. In S_n, set
$$\alpha' = \alpha + m,\ \beta' = \beta + m,\ n - m = n'.$$

Then
$$S_n = \frac{\alpha'(\alpha'+1)\cdots(\alpha'+n')}{\beta'(\beta'+1)\cdots(\beta'+n')}.$$

As $\alpha' > \beta' > 0$, S_n falls under case 1°.

Hence
$$\lim Q_n = \pm\infty,$$
the sign being that of R.

3°. $\alpha < \beta$. If $\alpha = 0, -1, -2, \cdots$, evidently $Q_n \doteq 0$.

Let
$$P_n = \frac{\beta(\beta+1)\cdots(\beta+n)}{\alpha(\alpha+1)\cdots(\alpha+n)}. \qquad \alpha \text{ not zero or a negative integer.}$$

Then P_n falls under cases 1° or 2°.

But
$$Q_n = \frac{1}{P_n}.$$

Hence
$$\lim Q_n = 0.$$

4°. $\alpha = \beta \neq 0, -1, -2, \cdots$. Here $Q_n = 1$.

Hence
$$\lim Q_n = 1.$$

Different Systems for Expressing Numbers

144. 1. Let a be a positive integer, and m any integer >1. Then we can give a the form
$$a = a_n m^n + a_{n-1} m^{n-1} + \cdots + a_0, \tag{1}$$
where
$$0 \leq a_\kappa \leq m - 1, \qquad \kappa = 0, 1, 2, \cdots$$
and n is a positive integer.

The number m is called the *base* of the system. The base being given, the numbers a_0, a_1, $\cdots$ a_n completely define the number a, and 1) may be written more shortly

$$a = a_n a_{n-1} \cdots a_0.$$

When $m = 10$, we have the decimal system. For example, we write

$$3 \cdot 10^4 + 1 \cdot 10^3 + 7 \cdot 10^2 + 7 \cdot 10 + 9$$

more shortly

$$31779.$$

When m is used as base, the numbers a are said to be expressed in an *m-adic system.*

2. Let $0 < \alpha < 1$. With α is associated a point on a right line L, whose distance from the origin of reference is α. To measure this distance, let us divide the unit interval into m equal parts, each of these parts into m equal parts, and so on. Then, as shown in 118, 120,

$$\alpha = \{l_1,\ l_2,\ l_3,\ \cdots\}, \qquad (2$$

where

$$l_1 = \frac{a_1}{m},$$

$$l_2 = l_1 + \frac{a_2}{m^2} = \frac{a_1}{m} + \frac{a_2}{m^2}, \qquad (3$$

$$l_3 = l_2 + \frac{a_3}{m^3} = \frac{a_1}{m} + \frac{a_2}{m^2} + \frac{a_3}{m^3},$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

The numbers l_1, l_2, $\cdots$ are completely determined when the numbers a_1, a_2, $\cdots$ are given. Thus α is determined when these latter numbers are given, and instead of representing α by the system of equations 2), 3), we may employ the shorter notation

$$\alpha = .a_1 a_2 a_3 \cdots$$

For example, let $\alpha = \frac{1}{3}$ and $m = 10$.

Here

$$a_1 = 3,\ a_2 = 3,\ a_3 = 3,\ \cdots$$

Hence

$$\tfrac{1}{3} = .3333 \cdots$$

which is the familiar decimal representation of $\frac{1}{3}$.

If we take $m = 5$, we get the representation

$$\tfrac{1}{3} = .1313131 \cdots$$

3. Thus every positive number can be written in the form

$$a_n a_{n-1} \cdots a_0 \cdot b_1 b_2 b_3 \cdots$$

where the a's and b's are $\geqq 0$ and $\leqq m-1$.

4. Certain numbers admit a double representation, in an m-adic system; viz. those numbers in which the b's, after a certain stage, all equal 0 or all equal $m-1$. In this case we have

$$\alpha = a_n a_{n-1} \cdots a_0 \cdot b_1 b_2 \cdots b_s 0000 \cdots \qquad (4$$

$$\alpha = a_n \cdots n_0 \cdot b_1 \cdots (b_s - 1)(m-1)(m-1) \cdots \qquad (5$$

For example, when $m = 10$,

$$23.5650000 \cdots$$

and

$$23.5649999 \cdots$$

represent the same number. In the future we shall suppose that all such numbers are represented by the form 4), which we shall call the *normal form.*

5. Numbers of the type

$$a_n a_{n-1} \cdots a_0 \cdot b_1 b_2 \cdots b\ \ 000 \cdots$$

all digits after b_ν being 0, are usually written more shortly, by omitting these zeros. Such numbers are said to admit a *finite representation.*

145. *The expression of a positive number N in normal form in an m-adic system is unique.*

1°. *Let N be an integer.* We show first that

$$m^n > a_0 + a_1 m + \cdots + a_{n-1} m^{n-1}. \qquad (1$$

This is obviously true for $n = 1$. We apply now the method of *complete induction.* Supposing 1) is true for $n = s$, we show it is true for $n = s + 1$. Let, then,

$$m^s > a_0 + a_1 m + \cdots + a_{s-1} m^{s-1}.$$

Then, since both numbers on the left and right are integers,

$$m^s - (a_0 + a_1 m + \cdots + a_{s-1} m^{s-1}) \geqq 1.$$

Hence

$$m^{s+1} - (a_0 + \cdots + a_{s-1} m^{s-1}) m \geqq m.$$

Subtracting an integer $b = 0, 1, \cdots m-1$, from both sides,

$$m^{s+1} - (b + a_0 m + a_1 m^2 + \cdots + a_{s-1} m^s) > 0.$$

Hence, changing the notation slightly,

$$m^{s+1} > a_0' + a_1' m + \cdots + a_s' m^s, \tag{2}$$

if the a' are $\leqq m-1$.

This established, let

$$M = a_0 + a_1 m + a_2 m^2 + \cdots + a_r m^r,$$

$$N = b_0 + b_1 m + b_2 m^2 + \cdots + b_s m^s,$$

where $a_r \neq 0$, $b_s \neq 0$.

Then 1) shows that if $r \gtrless s$, then $M \gtrless N$.

Hence, if M is to equal N, it is necessary that $r = s$.

If $r = s$, then 1) shows that $M \gtrless N$, according as $a_r \gtrless b_r$. Thus, if $M = N$, it is necessary that $a_r = b_r$. In this way we may continue, and so show that when $M = N$,

$$a_\kappa = b_\kappa, \quad \kappa = 0, 1, \cdots r.$$

2°. *Let* $0 < N < 1$.

Suppose

$$N = .a_1 a_2 \cdots a_{r-1} a_r \cdots$$

$$P = .a_1 a_2 \cdots a_{r-1} b_r \cdots$$

where $a_r \neq b_r$. To fix the ideas, let $a_r > b_r$. Then $N > P$.

For,

$$N \geqq \left(\frac{a_1}{m} + \cdots + \frac{a_{r-1}}{m^{r-1}}\right) + \frac{a_r}{m^r} = N_1 + N_2.$$

Since P is written in the normal form, there exists an $s \geqq r$, such that

$$b_s < m - 1.$$

Then

$$P \leqq N_1 + \frac{b_r}{m^r} + \cdots + \frac{b_s + 1}{m^s} = N_1 + P_2.$$

But since $a_r > b_r$,

$$N_2 > P_2.$$

Hence

$$N_1 + N_2 > N_1 + P_2,$$

and *a fortiori*,

$$N > P.$$

CHAPTER III

EXPONENTIALS AND LOGARITHMS

Rational Exponents

146. Having developed now the number system $\mathfrak{R}$ with sufficient detail, we shall in this and the subsequent chapters represent numbers in $\mathfrak{R}$ indifferently by Greek and Latin letters.

147. Up to the present we have defined the symbol

$$a^{\mu} \tag{1}$$

only for positive integral values of the exponent μ. We proceed to define it for any value of μ, supposing $a > 0$. We begin with rational values. The numbers 1) are then called *roots* or *radicals.*

148. 1. *Let* $a > 0$, *and let* n *be a positive integer. There exists one and only one positive number satisfying*

$$x^n = a. \tag{1}$$

Let (B, C) be a partition such that B contains all positive numbers b such that $b^n \leq a$.

Let ρ be the number which generates (B, C), 130, 2. By 130, 1, we can pick out of B a monotone increasing sequence $\{b_m\}$ and out of C a monotone decreasing sequence $\{c_m\}$ such that

$$\lim b_m = \lim c_m = \rho.$$

As

$$b_m^n \leq a \leq c_m^n,$$

we have, by 106, 2,

$$\lim_{m=\infty} b_m^n = a.$$

As, by 98,
$$\lim_{m=\infty} b_m^n = \rho^n,$$
we have
$$\rho^n = a. \qquad (2$$

Hence ρ satisfies 1).

There is but one positive solution of 1). For if
$$\sigma^n = a. \qquad (3$$
we have, from 2), 3),
$$\rho^n = \sigma^n.$$

Hence, by 75, 3,
$$\rho = \sigma.$$

2. We write
$$\rho = \sqrt[n]{a} = a^{\frac{1}{n}}.$$

3. When n is odd, 1) has only one solution in $\mathfrak{R}$, viz., $x = \sqrt[n]{a}$. When n is even, it has two and only two solutions in $\mathfrak{R}$, viz.,
$$\sqrt[n]{a}, \qquad -\sqrt[n]{a}.$$

149. 1. From the preceding we have readily:

Let $a < 0$. *Then*
$$x^n = a$$
has no solution if n is even, and if n is odd, it has one and only one solution, viz., $-\sqrt[n]{-a}$.

For brevity we often write, when n is odd and $a < 0$,
$$\sqrt[n]{a} \quad \text{for} \quad -\sqrt[n]{-a}.$$

When, however, $a > 0$ the radical $\sqrt[n]{a}$ shall always be a positive number.

2. *The equation* $x^n = 0$ *admits one and only one solution, viz.,* $x = 0$, *in* $\mathfrak{R}$. *We write*
$$\sqrt[n]{0} = 0^{\frac{1}{n}} = 0.$$

150. 1. *If m, n are positive integers, and $a > 0$, then*
$$(a^{\frac{1}{n}})^m = (a^m)^{\frac{1}{n}}. \qquad (1$$

Set
$$\rho = a^{\frac{1}{n}}.$$

Then

$$\rho^n = a,$$

and

$$\rho^m = (a^{\frac{1}{n}})^m; \tag{2}$$

also

$$a^m = \rho^{nm} = (\rho^m)^n. \tag{3}$$

The equation 3) shows that ρ^m is the positive solution of

$$x^n = a^m.$$

Hence, by definition,

$$\rho^m = (a^m)^{\frac{1}{n}}. \tag{4}$$

Comparing 2) and 4), we have 1).

2. We write

$$(a^{\frac{1}{n}})^m = (a^m)^{\frac{1}{n}} = a^{\frac{m}{n}}.$$

We have now the definition of

$$a^\mu, \qquad a > 0$$

for *positive rational exponents.*

151. Let μ be a *positive rational number.* We define the symbol $a^{-\mu}$ by the relation

$$a^{-\mu} = \frac{1}{a^\mu}.$$

We also set

$$a^0 = 1.$$

152. *Let r, s be rational numbers, and $a > 0$. Then*

$$a^r a^s = a^{r+s}. \tag{1}$$

This equation expresses the *addition theorem* for rational exponents. It is a generalization of 74, 2.

To fix the ideas, suppose $r, s > 0$. Let

$$r = \frac{l}{n}, \qquad s = \frac{m}{n},$$

where l, m, n are positive integers.

Let

$$\rho = a^r; \text{ then } \rho^n = a^l. \tag{2}$$

Let

$$\sigma = a^s; \text{ then } \sigma^n = a^m. \tag{3}$$

Multiplying 2), 3), we get

$$(\rho\sigma)^n = a^{l+m}.$$

This shows that $\rho \cdot \sigma$ is the positive root of

$$x^n = a^{l+m}.$$

Hence

$$\rho\sigma = a^{\frac{l+m}{n}}. \qquad (4$$

But 2), 3) also give

$$\rho\sigma = a^r a^s. \qquad (5$$

Comparing 4), 5), we get 1), since

$$\frac{l+m}{n} = r + s.$$

153. *Let μ be a rational number, and $a > 0$. Then*

$$a^\mu > 0.$$

For let $\mu = \frac{m}{n} > 0$; m, n positive integers.

We have, by 148,

$$a^{\frac{1}{n}} > 0.$$

Hence

$$a^{\frac{m}{n}} = a^{\frac{1}{n}} \cdot a^{\frac{1}{n}} \cdots a^{\frac{1}{n}} > 0. \quad m \text{ factors.}$$

If $\mu < 0$, let $\mu = -\nu$, $\nu > 0$.
Then

$$a^\mu = \frac{1}{a^\nu},$$

and as $a^\nu > 0$, so is $a^\mu > 0$.

If $\mu = 0$, we have $a^\mu = 1$, by 151.

154. *Let μ be a positive fraction and $a > 0$. Then*

$$a^\mu \gtreqless 1 \text{ according as } a \gtreqless 1.$$

Let $\mu = \frac{m}{n}$; m, n positive integers.

If $a > 1$, then $a^{\frac{1}{n}} > 1$.

For, suppose the contrary, *i.e.* let

$$a^{\frac{1}{n}} \leqq 1.$$

Raising to the nth powers, by 75, 2,

$$a \leq 1,$$

which is a contradiction. Hence

If $a > 1$, $$a^{\frac{1}{n}} > 1.$$

Hence $$a^{\frac{m}{n}} > 1, \text{ by 75, 2.}$$

The other cases are similarly treated.

155. *Let n be a positive integer and $a > 0$. Then $a^{\frac{1}{n}} \gtreqless a$ according as $a \lesseqgtr 1$.*

Let $a < 1$ and suppose $$a^{\frac{1}{n}} \leqq a.$$

Then, by 75, 2, $$a \leqq a^n. \qquad (1$$

But when $a < 1$, $$a^n < a,$$

by 75, 4. This contradicts 1).

Thus, when $a < 1$, $a^{\frac{1}{n}} > a$.

The other cases are easily treated now.

156. 1. *Let $a > 0$ and let μ be a positive fraction. If*

$$a^\mu > b > 0,$$

then $$a > b^{\frac{1}{\mu}}. \qquad (1$$

For, let $\mu = \dfrac{m}{n}$; m, n positive integers.

From $$a^{\frac{m}{n}} > b$$

follows, by 75, 2, $$a^m > b^m. \qquad (2$$

Suppose now 1) were not true, *i.e.* suppose

$$a \leqq b^{\frac{n}{m}}.$$

Then $$a^m \leqq b^n,$$

which contradicts 2).

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2. *Let $a > 0$ and let μ be a negative fraction.*

If

$$a^\mu > b > 0,$$

then

$$a < b^{\frac{1}{\mu}}. \qquad (3$$

We can set $\mu = -\nu$, $\nu > 0$.

Then

$$a^\mu = \frac{1}{a^\nu}.$$

This reduces 2 to 1.

157. *Let $\mu < \nu$ be two rational numbers; let $a > 0$.*

Then

$$a^\mu \lessgtr a^\nu \text{ according as } a \gtrless 1. \qquad (1$$

We can set

$$\mu = \frac{r}{t}, \quad \nu = \frac{s}{t},$$

where r, s, t are integers and $t > 0$.

To fix the ideas, let $a > 1$, and μ, $\nu > 0$.

Suppose for this case, 1) were not true, *i.e.* that

$$a^\mu \geqq a^\nu.$$

Then

$$a^r \geqq a^s, \text{ by 75, 2,}$$

which is absurd, since $r < s$.

Thus 1) is true for this case. In the same way we may treat the other case.

158. *Let μ be a rational number, and let $a_n > 0$, $n = 1, 2, \cdots$.*

If

$$\lim a_n = 1, \qquad (1$$

then

$$\lim a_n^\mu = 1. \qquad (2$$

To fix the ideas, let

$$\mu = \frac{r}{s}; \quad r, s \text{ positive integers.}$$

If

$$a_n < 1, \quad a_n < a_n^{\frac{1}{s}} < 1;$$

if

$$a_n > 1, \quad 1 < a_n^{\frac{1}{s}} < a_n,$$

by 154, 155. Thus in either case $a_n^{\frac{1}{s}}$ lies between 1 and a_n. Applying 107, we have, using 1),

$$\lim a_n^{\frac{1}{s}} = 1.$$

Since

$$a_n^{\frac{r}{s}} = a_n^{\frac{1}{s}} a_n^{\frac{1}{s}} \cdots a_n^{\frac{1}{s}}, \qquad r \textit{ factors}$$

we have, by 98,

$$\lim a_n^{\frac{r}{s}} = \lim a_n^{\mu} = 1.$$

The other cases are now easily treated.

159. *Let* μ *be a rational number; let*

$$\lim a_n = a > 0, \qquad a_n > 0.$$

Then

$$\lim a_n^{\mu} = a^{\mu}. \tag{1}$$

For,

$$a_n^{\mu} = a^{\mu} \left(\frac{a_n}{a}\right)^{\mu}. \tag{2}$$

But, by hypothesis,

$$\lim \frac{a_n}{a} = 1, \qquad \text{by 98.}$$

If we apply now 158 to 2), we get 1).

Irrational Exponents

160. *Let* $R = r_1, r_2, \cdots$ *be a sequence of rational numbers whose limit is* 0. *Then, if* $a > 0$,

$$\lim a^{r_n} = 1. \tag{1}$$

We show:

$$\epsilon > 0, \ m, \ |1 - a^{r_n}| < \epsilon, \ n > m, \tag{2}$$

which is the same as 1).

Let $a > 1$, $r_n > 0$; then, by 154,

$$a^{r_n} > 1.$$

Since $r_n > 0$ and as small as we please, n being sufficiently large, we can take m so large that

$$\frac{1}{r_n} > g, \qquad n > m,$$

however large the positive integer g be chosen.

But, by 94, 3),

$$(1 + \epsilon)^g > 1 + g\epsilon.$$

We can also take g so large that

$$1+g\epsilon>a.$$

Then

$$(1+\epsilon)^g>a. \tag{3}$$

On the other hand, by 157,

$$(1+\epsilon)^{\frac{1}{r_n}}>(1+\epsilon)^g. \tag{4}$$

Hence 3) and 4) give

$$(1+\epsilon)^{\frac{1}{r_n}}>a. \qquad n>m.$$

This gives, by 156, 1),

$$1+\epsilon>a^{r_n}.$$

Hence 2) holds in this case.

Let $a<1$, $r_n>0$. We set

$$a=\frac{1}{b};$$

then

$$b>1.$$

By the preceding case

$$b^{r_n}<1+\epsilon.$$

Hence

$$a^{r_n}=\frac{1}{b^{r_n}}>\frac{1}{1+\epsilon}>1-\epsilon,$$

by 89, 1). This gives

$$1-a^{r_n}<\epsilon.$$

Hence 2) holds in this case.

Let $r_n<0$. This case reduces to the case that $r_n>0$, by observing that

$$a^{r_n}=\left(\frac{1}{a}\right)^{-r_n}.$$

We consider now the case that the r_n do not all have one sign. We divide R into three sequences, R_0, R_+, R_-. In the first, we throw all $r_n=0$; in the second all $r_n>0$; in the third all $r_n<0$. Should any of these sequences contain only a finite number of elements, it can be neglected. For we have only to consider a partial sequence of R, obtained by omitting the terms of R up to a certain one.

Consider R_+. We have seen there exists in it an index m' such that 2) holds for every $n > m'$.

Similarly in R_-, there exists an index m'' such that 2) holds for every $n > m''$.

Consider finally R_0. As $r_n = 0$, 2) holds for every n of R_0.

Thus if m be taken $> m', m''$, 2) holds for every $n > m$ in R.

161. *Let* $A = a_1, a_2, \cdots$ *be a regular sequence of rational numbers, and let* $b > 0$. *Then*

$$b^{a_1}, b^{a_2}, \cdots \tag{1}$$

is regular.

We have to show:

$$\epsilon > 0, \ m, \ |b^{a_n} - b^{a_m}| < \epsilon, \qquad n > m. \tag{2}$$

Set

$$d_n = b^{a_n} - b^{a_m} = b^{a_m}(b^{a_n - a_m} - 1). \tag{3}$$

Since A is regular, we have

$$\delta > 0, \ m, \ |a_n - a_m| < \delta. \qquad n > m.$$

But if δ is taken small enough, by 160,

$$|b^{a_n - a_m} - 1| < \eta, \tag{4}$$

where $\eta > 0$ is arbitrarily small.

Since A is regular, there exist, by 65, 5, two rational numbers, Q, R, such that

$$Q < a_n < R. \qquad n = 1, 2, \cdots \tag{5}$$

Then 3) gives, by 4) and 5),

$$\begin{aligned} |d_n| &< b^R\eta, \text{ if } b > 1; \\ &< b^Q\eta, \text{ if } b < 1, \text{ by 157.} \end{aligned} \tag{6}$$

If $b > 1$, we take $\eta = \frac{\epsilon}{b^R}$;

if $b < 1$, we take $\eta = \frac{\epsilon}{b^Q}$.

Then in either case 2) holds.

The case that $b = 1$ requires no demonstration.

162. *Let $a_1, a_2, \cdots$ and $\alpha_1, \alpha_2, \cdots$ be two sequences of rational numbers having the same limit. Then, if $b > 0$,*

$$\lim b^{a_n} = \lim b^{\alpha_n}. \qquad (1$$

By 161, both limits in 1) exist.

Let
$$d_n = b^{a_n} - b^{\alpha_n} = b^{a_n}(1 - b^{\alpha_n - a_n}). \qquad (2$$

We have only to show that

$$\lim d_n = 0. \qquad (3$$

But
$$\lim (a_n - \alpha_n) = 0.$$

Hence, by 160,
$$\lim (1 - b^{\alpha_n - a_n}) = 0. \qquad (4$$

Hence, 2), 4) give 3).

163. We are now in the position to define *irrational exponents.* Let

$$\mu = (r_1, r_2, \cdots)$$

be a representation of μ. We say

$$a^{\mu} = \lim a^{r_n}. \qquad (1$$

By 161, the limit on the right of 1) exists; by 162, it is the same whatever representation of μ is taken.

164. 1. *Let $r_1, r_2, \cdots$ be a sequence of rational numbers having a rational limit r. Then,* if $b > 0$,

$$\lim b^{r_n} = b^{r}. \qquad (1$$

In fact, the sequence

$$r'_1, r'_2, r'_3, \cdots; \qquad r'_n = r, \qquad n = 1, 2, \cdots$$

has r as limit.

By 162,
$$\lim b^{r_n} = \lim b^{r'_n}. \qquad (2$$

But
$$\lim b^{r'_n} = \lim b^{r} = b^{r}.$$

This in 2) gives 1).

2. The object of 1 is to show that the definition of a^{μ} given in 163 does not conflict with that given in 150, 151, in case μ is rational.

165. 1. *Let* μ *be an arbitrary number, and* r, s, *two rational numbers, such that* $r < \mu < s$. *Then for* $b > 0$,

$$b^r \gtrless b^\mu \gtrless b^s, \text{ according as } b \lessgtr 1. \qquad (1$$

For, let

$$\mu = (m_1, m_2, \cdots),$$

the m's being rational.

Then, by 105, 1,

$$r < m_n < s. \qquad n > \nu.$$

Hence, by 157,

$$b^r < b^{m_n} < b^s, \quad \text{if} \quad b > 1.$$

Then passing to the limit, by 106, 1,

$$b^r \leq b^\mu \leq b^s. \qquad (2$$

Here the equality sign must be suppressed. For, let r' be another rational number such that

$$r < r' < \mu.$$

Then, as in 2),

$$b^r \leq b^\mu. \qquad (3$$

But

$$b^r < b^{r'}, \qquad (4$$

by 157. From 3), 4) we have

$$b^r < b^\mu.$$

Thus the equality sign in 2) must be suppressed.

The truth of 1), when $b < 1$, follows in a similar manner.

2. As a corollary of 1, we have:

Let $a > 0$; *then* a^μ *vanishes for no value of* μ.

In fact, the relation 1 shows that a^μ always lies between two positive numbers, by 153.

166. 1. The properties given in the preceding articles for rational exponents hold for irrational exponents also. We illustrate the demonstration in a few cases.

Let $\lambda < \mu$, *and* $b > 0$. *Then*

$$b^\lambda \lessgtr b^\mu, \text{ according as } b \gtrless 1. \qquad (1$$

To fix the ideas, suppose $b > 1$. Let r be a rational number, such that

$$\lambda < r < \mu.$$

Then, by 165,

$$b^\lambda < b^r < b^\mu.$$

Hence

$$b^\lambda < b^\mu.$$

2. As corollary we have:

If $b > 1$, *we conclude from*

$$b^\lambda \gtreqless b^\mu,$$

that

$$\lambda \gtreqless \mu;$$

whereas, if $0 < b < 1$, *we conclude that*

$$\lambda \lesseqgtr \mu.$$

3. In 1) let $\lambda = 0$. Since

$$b^0 = 1$$

we have:

Let

$$\mu > 0, \text{ and } b > 0; \text{ then}$$

$$b^\mu \gtreqless 1, \text{ according as } b \gtreqless 1.$$

167. $$b^{-\alpha} = \frac{1}{b^\alpha}. \qquad b > 0.$$

For, let

$$\alpha = (a_1 a_2 \cdots).$$

Then

$$-\alpha = (-a_1, -a_2, \cdots), \text{ by } 71, 3.$$

Since, by 151,

$$b^{-a_n} = \frac{1}{b^{a_n}},$$

we have

$$b^{-\alpha} = \lim b^{-a_n} = \frac{1}{\lim b^{a_n}} = \frac{1}{b^\alpha},$$

since $b^\alpha \neq 0$, by 165, 2.

168. *If* $a>0$,

$$a^\lambda a^\mu = a^{\lambda+\mu}.$$

This is the *addition theorem* for any exponents, and is a generalization of 152.

Let

$$\lambda = \lim \lambda_n, \quad \mu = \lim \mu_n.$$

Then

$$a^\lambda = \lim a^{\lambda_n}, \quad a^\mu = \lim a^{\mu_n}.$$

Hence

$$a^\lambda a^\mu = \lim a^{\lambda_n} \lim a^{\mu_n} = \lim a^{\lambda_n} a^{\mu_n}$$

$$= \lim a^{\lambda_n+\mu_n}, \text{ by 152,}$$

$$= a^{\lambda+\mu}.$$

169. *Let* $\lambda_1, \lambda_2, \cdots$ *be a sequence whose limit is* $+\infty$.

If $a>0$,

$$\lim a^{\lambda_n} = \begin{cases} +\infty, & \text{if } a>1, \\ 0, & \text{if } a<1. \end{cases}$$

For, *let* $a>1$.

Let l_n be the greatest integer in λ_n.

Since $\quad \lim \lambda_n = +\infty, \qquad \lim l_n = +\infty.$

Then, by 140,

$$\lim a^{l_n} = +\infty.$$

As

$$a^{\lambda_n} \overline{\overline{>}} a^{l_n},$$

$$\lim a^{\lambda_n} = +\infty,$$

by 138.

Let $a<1$.

Set $\quad b = \dfrac{1}{a}. \qquad$ Then $b>1$.

The demonstration follows now at once.

170. *Let* $a_1, a_2, \cdots$ *be a sequence of positive numbers whose limit is* 1.

Then

$$\lim a_n{}^\lambda = 1.$$

Let r, s be rational numbers, such that

$$r < \lambda < s.$$

Then, by 166, 1,

$$a_n^r \leq a_n^\lambda \leq a_n^s, \qquad a_n \geq 1; \qquad (1$$

$$a_n^s < a_n^\lambda < a_n^r, \qquad a_n < 1. \qquad (2$$

Let us apply now 107. Since, by 158,

$$\lim a_n^r = \lim a_n^s = 1,$$

we have, from 1), 2),

$$\lim a_n^\lambda = 1.$$

171. *Let* $a_1, a_2, \cdots$ *be a sequence of positive numbers whose limit is* $\alpha > 0$. *Then*

$$\lim a_n^\lambda = \alpha^\lambda. \qquad (1$$

For,

$$a_n^\lambda = \alpha^\lambda \left(\frac{a_n}{\alpha}\right)^\lambda. \qquad (2$$

Since, by 170,

$$\lim \frac{a_n}{\alpha} = 1,$$

1) follows from 2) at once.

172. *Let* $\lambda_1, \lambda_2, \cdots$ *be a sequence whose limit is* λ. *If* $a > 0$,

$$\lim a^{\lambda_n} = a^\lambda. \qquad (1$$

For, let

$$r_1, \ r_2, \ \cdots, \qquad s_1, \ s_2, \ \cdots$$

be two sequences of rational numbers whose limits are λ, and such that

$$r_n < \lambda_n < s_n, \qquad n = 1, 2, \cdots$$

To fix the ideas, let $a > 1$; then, by 166,

$$a^{r_n} < a^{\lambda_n} < a^{s_n}. \qquad (2$$

By 162,

$$\lim a^{r_n} = \lim a^{s_n}.$$

The application of 107 to 2) gives 1).

The case that $a \leqq 1$ is now readily treated.

Logarithms

173. *Let $a, b > 0$, and $b \neq 1$. The equation*

$$b^x = a \tag{1}$$

has one, and only one, solution.

To fix the ideas, let $b > 1$. We form a partition (C, D) in which C contains all numbers c, such that

$$b^c \leq a;$$

while D contains all numbers d, such that

$$b^d > a.$$

This separation of the numbers of $\mathfrak{R}$ into the classes C, D is indeed a partition. For every number of C is $<$ any number of D.

In fact, from

$$b^c < b^d,$$

follows, by 166, 2,

$$c < d.$$

Let ξ be the number which generates (C, D); let

$$c_1 \leq c_2 \leq \cdots$$

$$d_1 \geq d_2 \geq \cdots$$

be the monotone sequences of 130, whose common limit is ξ.

Then

$$b^\xi = a. \tag{2}$$

For, by 171,

$$\lim b^{c_n} = \lim b^{d_n} = b^\xi. \tag{3}$$

On the other hand,

$$b^{c_n} < a < b^{d_n}. \tag{4}$$

From 3), 4) we have, by 106, 2,

$$\lim b^{c_n} = a. \tag{5}$$

From 3), 5) we have 2).

The equation 2) shows that ξ is a solution 1). Let η be also a solution, so that

$$b^\eta = a. \tag{6}$$

From 2), 6) we have

$$b^\xi = b^\eta.$$

Hence, from 166, 2,

$$\xi = \eta.$$

174. 1. As we have just seen, the equation

$$b^x = a, \qquad a > 0; \qquad b > 0 \textit{ and } \neq 1,$$

admits one, and only one, solution. This uniquely determined number ξ, we call the *logarithm of* a, *the base being* b; and write

$$\xi = \log_b a,$$

or when we do not care to indicate the base,

$$\xi = \log a.$$

2. We shall suppose, once for all, that the base b is $\neq 1$; also that the numbers whose logarithms we are considering are > 0.

3. *From*

$$\log u = \log v,$$

follows

$$u = v.$$

The demonstration is obvious.

175. $$\log ab = \log a + \log b.$$

This is the *addition theorem of logarithms.*

Let the base be c. If

$$\alpha = \log a, \quad \beta = \log b,$$

then

$$c^\alpha = a, \quad c^\beta = b.$$

Multiplying, we have

$$c^\alpha c^\beta = c^{\alpha+\beta} = ab.$$

From the equation

$$c^{\alpha+\beta} = ab,$$

we have

$$\log ab = \alpha + \beta = \log a + \log b.$$

176. By using the properties of exponentials we may deduce in a similar manner all the ordinary properties of logarithms. As this presents nothing of interest, we pass on. We note, however, the following important relation.

Let $a > 0$, *and* b *be the base of our logarithms. Then*

$$a^\mu = b^{\mu \log a}. \tag{1}$$

For, by definition,

$$b^{\log a^\mu} = a^\mu. \qquad (2$$

But

$$\log a^\mu = \mu \log a.$$

This in 2) gives 1).

177. *Let* $a_1, a_2, \cdots$ *be a sequence whose limit is* 1. *Then*

$$\lim \log a_n = 0. \qquad (1$$

To fix the ideas, let b, the base of our logarithms, be >1. Let $\epsilon > 0$, then, by 166, 3,

$$b^\epsilon > 1. \qquad (2$$

Hence

$$\delta = 1 - \frac{1}{b^\epsilon} > 0. \qquad (3$$

Since

$$\lim a_n = 1,$$

we have

$$\delta > 0, \quad m, \quad -\delta < a_n - 1 < \delta, \quad n > m;$$

which gives

$$1 - \delta < a_n < 1 + \delta. \qquad (4$$

From 3) we have

$$1 - \delta = \frac{1}{b^\epsilon}.$$

This in 4) gives

$$a_n > \frac{1}{b^\epsilon}. \qquad (5$$

On the other hand,

$$\delta = \frac{b^\epsilon - 1}{b^\epsilon} < b^\epsilon - 1,$$

by 3), 2).

This gives

$$1 + \delta < b^\epsilon. \qquad (6$$

Then 6) and 4) give

$$a_n < b^\epsilon. \qquad (7$$

From 5), 7) we have finally

$$b^{-\epsilon} < a_n < b^\epsilon.$$

This may be written, by 176, 1),

$$b^{-\epsilon} < b^{\log a_n} < b^\epsilon.$$

Hence, by 166, 2,

$$-\epsilon < \log a_n < \epsilon, \qquad n > m,$$

which is another form of 1).

178. *Let* $\lim a_n = \alpha > 0$. *Then*

$$\lim \log a_n = \log \alpha. \qquad a_n > 0. \tag{1}$$

For,

$$a_n = \alpha \cdot \frac{a_n}{\alpha}.$$

Hence

$$\log a_n = \log \alpha + \log \frac{a_n}{\alpha}. \tag{2}$$

As

$$\lim \frac{a_n}{\alpha} = 1,$$

we need only to apply 177 to 2) to get 1).

179. *Let* $a_1 a_2 \cdots$ *be a sequence whose limit is* $+\infty$. *Then*

$$\lim \log_b a_n = \begin{cases} +\infty \text{ if } b > 1, \\ 0 \quad \text{if } b < 1. \end{cases}$$

Let $b > 1$. Let m_n be an integer, such that

$$b^{m_n} \leq a_n \leq b^{m_n+1}.$$

Then, by 176,

$$\log a_n > m_n. \tag{1}$$

But

$$\lim m_n = +\infty,$$

since

$$\lim a_n = +\infty.$$

Hence, by 138, using 1),

$$\lim \log a_n = +\infty.$$

The case that $b < 1$ follows at once now.

Some Theorems on Limits

180. *Let* $A = a_1, a_2, \cdots$ *be any sequence, such however that its limit is* $\pm\infty$ *when it is not limited; let* $B = b_1, b_2, \cdots$ *be an increasing sequence whose limit is* $+\infty$. *If*

$$l = \lim \frac{a_{n+1} - a_n}{b_{n+1} - b_n} \tag{1}$$

is finite or infinite, then

$$\lim \frac{a_n}{b_n} = l. \tag{2}$$

Proof. 1°. *l finite.*

Set

$$q_n = \frac{a_{n+1} - a_n}{b_{n+1} - b_n}, \quad q_{n,p} = \frac{a_{n+p} - a_n}{b_{n+p} - b_n}.$$

From 1) we have:

$$\delta > 0, \quad m, \quad |l - q_n| < \delta, \quad n \geqq m.$$

Hence

$$l - \delta < q_m \quad < l + \delta,$$

$$l - \delta < q_{m+1} \quad < l + \delta,$$

$$\vdots$$

$$l - \delta < q_{m+p-1} < l + \delta.$$

To these inequalities apply 93, setting the γ's equal 1. Then

$$l - \delta < q_{m,p} < l + \delta,$$

or

$$q_{m,p} - \delta < l < q_{m,p} + \delta. \tag{3}$$

If A is limited;

$$\lim_{p=\infty} q_{m,p} = 0,$$

since, by hypothesis, $b_n \doteq +\infty$.

In 3, pass to the limit $p = \infty$; we get

$$-\delta \leq l \leq \delta.$$

Hence

$$l = 0.$$

But on the supposition that A is limited,

$$\lim \frac{a_n}{b_n} = 0.$$

Thus 2) holds in this case.

If A is not limited;

$\delta > 0$ being small at pleasure, and m fixed, we have, by 92,

$$\delta > 0, \quad p_0, \quad \frac{1 - \dfrac{b_m}{b_{m+p}}}{1 - \dfrac{a_m}{a_{m+p}}} = 1 + \delta_1; \quad |\delta_1| < \delta, \quad p > p_0. \tag{4}$$

Now
$$q_{m,p} = \frac{a_{m+p}\left(1 - \frac{a_m}{a_{m+p}}\right)}{b_{m+p}\left(1 - \frac{b_m}{b_{m+p}}\right)}. \tag{5}$$

Also, by 3),
$$q_{m,p} = l + \delta'. \qquad |\delta'| < \delta. \tag{6}$$

From 5) we get, using 4), 6),
$$\frac{a_{m+p}}{b_{m+p}} = (l + \delta')(1 + \delta_1)$$
$$= l + l\delta_1 + \delta' + \delta_1\delta'.$$

Hence
$$\frac{a_{m+p}}{b_{m+p}} - l = l\delta_1 + \delta' + \delta_1\delta',$$

and
$$\left| l - \frac{a_{m+p}}{b_{m+p}} \right| < \delta(|l| + 1 + \delta)$$
$$< \delta(|l| + 2), \tag{7}$$

supposing $\delta < 1$.

If now we take
$$\delta < \frac{\epsilon}{2 + |l|},$$

7) gives 2).

2°. *l infinite.* To fix the ideas, suppose $l = +\infty$. Then
$$g > 0, \ m, \ q_n > g. \qquad n \geq m.$$

Hence
$$q_m, \ q_{m+1}, \cdots q_{m+p-1} > g.$$

Applying 93, we get
$$q_{m,p} > g. \qquad p = 1, 2, \cdots$$

This shows that
$$\lim a_n = +\infty.$$

Then 5) shows that, taking η such that $0 < \eta < 1$,
$$q_{m,p} = \frac{a_{m+p}}{b_{m+p}}(1 + \eta'); \ |\eta'| < \eta, \ p > p_0,$$

by 92. Hence
$$\frac{a_{m+p}}{b_{m+p}} = \frac{q_{m,p}}{1 + \eta'} > \frac{g}{1 + \eta}.$$

If we take

$$g > (1+\eta) G,$$

where G is arbitrarily large, we have

$$\frac{a_n}{b_n} > G. \qquad n > m + p.$$

This proves 2) for this case.

181. *Let* $a_1, a_2, \cdots$ *be a sequence whose limit, finite or infinite, is* α. *Then*

$$\lim \frac{a_1 + a_2 + \cdots + a_n}{n} = \alpha.$$

Let

$$A_n = a_1 + \cdots + a_n.$$

Then

$$a_n = \frac{A_n - A_{n-1}}{n - (n-1)}.$$

Hence, by 180,

$$\lim \frac{A_n}{n} = \alpha.$$

182. *Let* $a_1, a_2, \cdots$ *be a sequence whose terms are positive, and whose limit, finite or infinite, is* $\alpha > 0$.
Then

$$\lim \sqrt[n]{a_1 a_2 \cdots a_n} = \alpha. \tag{1}$$

1°. *α finite.* Consider the auxiliary sequence

$$\log a_1, \log a_2, \cdots; \text{ base} > 1.$$

By 178,

$$\lim \log a_n = \log \alpha. \tag{2}$$

By 181 and 2),

$$\lim \frac{1}{n}(\log a_1 + \cdots + \log a_n) = \log \alpha. \tag{3}$$

But

$$\frac{1}{n}(\log a_1 + \cdots + \log a_n) = \log \sqrt[n]{a_1 a_2 \cdots a_n}. \tag{4}$$

From 3), 4) we have 1).

2°. *α infinite.* To fix the ideas, let $\alpha = +\infty$.
By 179,

$$\lim \log a_n = +\infty.$$

By 181,

$$\lim \log \sqrt[n]{a_1 a_2 \cdots a_n} = \alpha = +\infty.$$

Thus

$$\lim \sqrt[n]{a_1 \cdots a_n} = +\infty.$$

Hence 1) is true in this case.

183. *Let $a_1, a_2, \cdots$ be a sequence of positive numbers.*
Let

$$lim \frac{a_n}{a_{n-1}} = \alpha \neq 0\,; \text{ *finite or infinite.*}$$

Then

$$\lim \sqrt[n]{a_n} = \alpha.$$

For,

$$\sqrt[n]{a_n} = \sqrt[n]{\frac{a_n}{a_{n-1}} \cdot \frac{a_{n-1}}{a_{n-2}} \cdots \frac{a_2}{a_1} \cdot a_1}.$$

Apply now 182.

184. *Let $a_1, a_2, \cdots$ be a sequence whose limit is* 0.
Let $b_1, b_2, \cdots$ be a decreasing sequence whose limit is 0.
Let

$$\lim \frac{a_n - a_{n-1}}{b_n - b_{n-1}} = l\,; \textit{ finite or infinite.}$$

Then

$$\lim \frac{a_n}{b_n} = l. \qquad (1$$

1°. *l finite.* As in 180, we have

$$\epsilon > 0,\ m',\ \left| \frac{a_m - a_{m+p}}{b_m - b_{m+p}} - l \right| < \frac{\epsilon}{2};\ p = 1, 2, \cdots \qquad m > m'. \qquad (2$$

Since by 92,

$$\lim_{p=\infty} \frac{a_m - a_{m+p}}{b_m - b_{m+p}} = \frac{a_m}{b_m},$$

we have

$$\epsilon,\ p_0,\ \left| \frac{a_m - a_{m+p}}{b_m - b_{m+p}} - \frac{a_m}{b_m} \right| < \frac{\epsilon}{2}; \qquad p > p_0. \qquad (3$$

Adding 2), 3), we get

$$\left| \frac{a_m}{b_m} - l \right| < \epsilon; \qquad m > m'.$$

2°. *l infinite.* Let $l = +\infty$.

Then

$$G > 0, \ m', \ q_{m,p} = \frac{a_m - a_{m+p}}{b_m - b_{m+p}} > G; \ p = 1, 2, \cdots \qquad m > m'.$$

But for sufficiently large p,

$$q_{m,p} = \frac{a_m}{b_m} + \delta_1; \ |\delta_1| < \delta, \text{ by } 92.$$

Hence

$$\frac{a_m}{b_m} > G - \delta = g.$$

Hence

$$\frac{a_m}{b_m} > g; \qquad m > m'.$$

and g is large at pleasure, since G is.

185. EXAMPLES

1.
$$\lim_{n=\infty} \frac{\log n}{n} = 0.$$

For,
$$\frac{\log n - \log (n-1)}{n - (n-1)} = \log \frac{n}{n-1} \doteq 0.$$

2.
$$\lim_{n=\infty} \frac{e^n}{n} = +\infty.$$

For,
$$\frac{e^n - e^{n-1}}{n-(n-1)} = e^n \left(1 - \frac{1}{e}\right) \doteq +\infty.$$

3.
$$\lim_{n=\infty} \sqrt[n]{n} = 1.$$

For,
$$\frac{n}{n-1} \doteq 1.$$

4.
$$\lim_{n=\infty} \sqrt[n]{n!} = +\infty.$$

For,
$$\frac{n!}{\overline{n-1}!} = n \doteq +\infty.$$

CHAPTER IV

THE ELEMENTARY FUNCTIONS. NOTION OF A FUNCTION IN GENERAL

FUNCTIONS OF ONE VARIABLE

Definitions

186. The functions of elementary mathematics are the following:

Integral rational functions.
Rational functions.
Algebraic functions.
Circular functions.
Exponential functions.
Inverse circular functions.
Logarithmic functions.

The reader is already familiar with the simpler properties of these functions, which we may call the *elementary functions*. We wish, however, to restate some of them for the sake of clearness.

187. In *applied* mathematics we deal with a great variety of quantities, as length, area, mass, time, energy, electromotive force, entropy, etc. In a given problem some of these quantities vary, others are fixed or constant.

The measures of these quantities are numbers.

In certain parts of *pure* mathematics we study the relations between certain sets of numbers without reference to any physical or geometrical quantities they may measure. In either case we find it convenient to employ certain letters or symbols to which we assign one or more numbers, or as we say, numerical values.

A symbol which has only one value in a given problem is a *constant*.

A symbol which takes on more than one value, in general an infinity of values, is a *variable*.

188. The set of values a variable takes on is called the *domain of the variable.*

It is often convenient to represent the values of a variable by points on a right line called the *axis* of the variable, as explained in 123. The domain of a variable may embrace all the numbers in $\mathfrak{R}$, or, as is more often the case, only a part of these numbers. Very frequently the domain is, speaking geometrically, an interval; *i.e.* the variable x takes on all values satisfying the relation

$$a \leq x \leq b.$$

Such an interval we shall represent by the symbol

$$(a, b).$$

Frequently one or both the end points a, b are excluded. Then we use the symbols

$$(a^*, b) \text{ for } a < x \leq b;$$

$$(a, b^*) \text{ for } a \leq x < b;$$

$$(a^*, b^*) \text{ for } a < x < b.$$

Similarly,

$$(a, +\infty) \text{ includes all } x \geqq a;$$

$$(-\infty, a) \text{ includes all } x \leqq a;$$

$$(-\infty, +\infty) \text{ includes all } x \text{ in } \mathfrak{R}.$$

A point of the interval (a, b) which is not an end point is *within* the interval.

189. Let x be a variable, whose domain call D. Let a law be given which assigns for each value of x in D one or more values to y. We say *y is a function of x*, and write

$$y = f(x), \text{ or } y = \phi(x), \text{ etc.}$$

If y has only one value assigned to it for each value of x in D, we say y is a *one-valued function*, otherwise y is *many-valued.* The variable x is called the *independent variable* or *argument;* y is called the *dependent variable*

We must note, however, that y may be a constant.

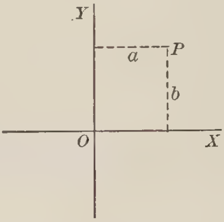
The domain of the independent variable x which enters in the law defining a function $f(x)$ is also called the *domain of definition of the function.*

The above very general definition of a function is due to *Dirichlet.*

190. The reader is already familiar with the graphical representation of a function, by the aid of two rectangular axes.

Let

$$y = f(x)$$



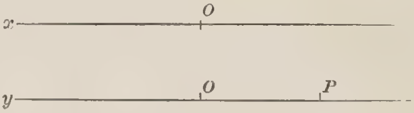
be a given function whose domain of definition call D.

The graphical representation of D is a set of points on the x-axis.

Let a be a value of x to which corresponds the value b of y. The point P in the figure whose coördinates are a, b represents the value of the function for $x = a$.

As x runs over the values of its domain D, the point P runs over a set of points, which we call the graph of $f(x)$.

191. 1. Another representation of a function is the following: We take two axes as in the figure; one for x, one for y. In this representation, the graph of $f(x)$ is a set of points on the y-axis.



2. The reader will observe this important difference between the two modes of representation just given. In the first we know for each point P of the graph the corresponding values of *both* x and y. In the second mode of representation, we do not know in general the value of x corresponding to a point P of the graph, and conversely. In spite of this deficiency, we shall find that this second representation is extremely useful. This is especially the case when we come to consider functions of n variables.

192. Ex. 1. Let D be given by

$$0 \leq x \leq 1;$$

while y is given by

$$y = x^2.$$

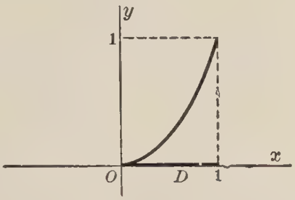


FIG. 1.

The graph of y in the first mode of representation is the arc of the parabola given in Fig. 1. The domain of definition D is the segment (0, 1) on the x-axis, drawn heavy in the figure.

In the second mode of representation the graph of y is the segment marked heavy on the y-axis (Fig. 2).

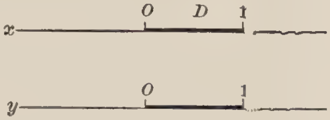


FIG. 2.

193. Ex. 2. As in Ex. 1, let

$$y = x^2.$$

Let, however, the domain of definition D embrace only the values of

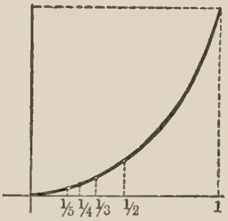
$$x = 1,\ \tfrac{1}{2},\ \tfrac{1}{3},\ \tfrac{1}{4},\ \cdots$$

In the first representation the graph of y is a set of points lying on the arc of the parabola of Ex. 1.

In the second representation it is the set of points

$$1,\ \tfrac{1}{4},\ \tfrac{1}{9},\ \tfrac{1}{16},\ \cdots$$

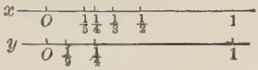
on the y-axis.



In both modes of representation, the representation of D is the set of points

$$1,\ \tfrac{1}{2},\ \tfrac{1}{3},\ \tfrac{1}{4},\ \tfrac{1}{5},\ \cdots$$

on the x-axis.



Integral Rational Functions

194. These functions are of the type

$$y = a_0 + a_1 x + a_2 x^2 + \cdots + a_m x^m, \qquad (1$$

where the a's are constants, and m is a positive integer or 0.

Such functions are called *polynomials* in algebra.

The number m is called the *degree* of the polynomial y.

When $m=1$, we have

$$y=a_0+a_1x. \tag{2}$$

The graph of 2) is a straight line. For this reason an integral rational function of the first degree is called *linear*.

When $m=0$ or when $a_1=0$ in 2), we have

$$y=a_0, \text{ a constant.} \tag{3}$$

We still say y is a function of x. In fact, 3) states that for each value of x, the corresponding value of y is a_0.

The graph of 3) is a line parallel to the x-axis. Since the equation 1) assigns to y a value for each value of x, the domain of definition of y embraces all numbers of $\mathfrak{R}$. Speaking geometrically, as we often shall in the future, it includes all the points of the x axis.

Since 1) assigns only one value to y for each value of x, y is a one-valued function.

195. In algebra we learn that if a polynomial

$$P_m=a_0+a_1x+\cdots a_mx^m \tag{1}$$

vanishes for $x=\alpha$, we can write

$$P=(x-\alpha)P_{m-1},$$

where P_{m-1} is a polynomial of degree $m-1$. We learn also in algebra that a polynomial of degree m cannot vanish more than m times, without being identically 0, in which case all the coefficients in 1) are 0.

Should 1) vanish for

$$x=\alpha_1,\ \alpha_2,\ \cdots\ \alpha_m, \tag{2}$$

we have

$$P_m=a_m(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_m).$$

The numbers 2) are called *roots* or *zeros* of P_m.

Rational Functions

196. The quotient of two integral rational functions of x is called a *rational function.*

Their general type is given by

$$y=\frac{a_0+a_1x+\cdots+a_mx^m}{b_0+b_1x+\cdots+b_nx^n}, \tag{1}$$

where the a's and b's are constants and m, n are positive integers or 0.

The expression 1) involves division by zero for those values of x for which the denominator vanishes. The domain of definition D of a rational function includes therefore all points on the x-axis except the zeros of the denominator. These zeros we shall call the *poles* of y. Since 1) assigns only one value to y for each point of D, a rational function of x is one-valued.

The degree of y is the greater of the two integers m, n; supposing, of course, that $a_m, b_n \neq 0$.

When y is of the first degree, it is called linear. The type of a linear rational function is, therefore,

$$y=\frac{a_0+a_1x}{b_0+b_1x}.$$

The rational function includes the integral rational function as a special case.

In fact, let the numerator be divisible by the denominator, then 1) reduces to a polynomial. This takes place in particular when the denominator reduces to a constant.

Algebraic Functions

197. We say y is an algebraic function of x when it satisfies an equation of the type

$$y^n+R_1y^{n-1}+R_2y^{n-2}+\cdots+R_{n-1}y+R_n=0, \tag{1}$$

where n is a positive integer, and the R's are rational functions of x.

The *degree* of y is n.

Let us give to x a definite value $x = a$. If a is a pole of any of the R's, the equation 1) has no meaning for this point, and it does not lie in the domain of definition of y.

Suppose a is not a pole of any of the R's.

Then each R_m takes on a constant value, say A_m, and 1) goes over into

$$y^n + A_1 y^{n-1} + \cdots + A_{n-1} y + A_n = 0, \qquad (2$$

an equation with constant coefficients of degree n.

Equation 2) may have no real roots. In this case a does not belong to the domain of definition of y. On the other hand, 2) cannot have more than n real roots for $x = a$.

These considerations show that y is at most an n-valued function whose domain of definition embraces all or only a part of the x-axis.

If we clear of fractions in 1), we may write it

$$P_0 y^n + P_1 y^{n-1} + \cdots + P_{n-1} y + P_n = 0, \qquad (3$$

where now the coefficients of y are polynomials in x. Evidently either 1) or 3) may be used as definition of an algebraic function.

198. The algebraic functions include the rational functions as a special case.

For, if $n = 1$, the equation 1) of 197 takes on the form

$$y + R_1 = 0,$$

or

$$y = - R_1.$$

But $- R_1$ is any rational function.

199. An expression which can be formed from x and certain constants by the four rational operations and the extraction of roots, each repeated a finite number of times, is called an *explicit algebraic function.*

Such a function is

$$y = a - \frac{b}{x^2} + \sqrt[3]{\frac{a + bx^4}{b + \sqrt{c + dx^3}}} + \sqrt[5]{\frac{1}{1 - x^2} + \sqrt{a + bx}}. \qquad (1$$

Obviously 1) can be obtained from

$$x,\ a,\ b,\ c,\ d$$

by the aid of the five operations, addition, subtraction, multiplication, division, and the extraction of roots, each repeated only a finite number of times.

It is known that every explicit algebraic function y satisfies an equation of the type 197, 1). Hence every explicit algebraic function is an algebraic function, by 197.

The converse is, however, not true; every algebraic function y cannot be brought into the form of an explicit algebraic function.

This is due to the fact that equations of degree $n > 4$ cannot, in general, be solved by the extraction of roots, or, as we say, do not admit of an algebraic solution.

200. All functions which are not algebraic functions are called *transcendental functions.*

The terms *algebraic* and *transcendental* may also be applied to the numbers of $\mathfrak{R}$.

Any number α which satisfies an equation of the type

$$x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_{n-1} x + a_n = 0, \tag*{(1}$$

where n is a positive integer, and the a's are rational numbers, is called an *algebraic number.* All other numbers of $\mathfrak{R}$ are *transcendental numbers.*

When $n = 1$, the equation 1) defines a rational number; the rational numbers are special cases of algebraic numbers.

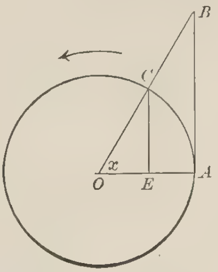
Circular Functions

201. As the reader already knows, the circular functions may be defined as the lengths of certain lines connected with a circle of unit radius.

Thus, in the figure

$$\sin x = CE, \ \cos x = OE, \ \tan x = AB,$$

etc. We have shown in Chapter II how the rectilinear segments AB, CE, etc., are measured. It has not yet been shown, however, how to measure arcs of a circle, *i.e.* how to each arc as AC, a number x may be attached, as its measure.



This will be given later. No inconvenience can arise if we assume here a knowledge of this theory inasmuch as the reader is perfectly conversant with its results, which are all we need for the present.

Arcs measured in the direction of the arrow are positive; those measured in the opposite direction are negative.

If we suppose the point C to move around the circle in a positive direction starting from a fixed point A as point of reference, it has described an arc whose measure is 2π,

$$\pi = 3.14159265 \cdots$$

when it reaches A again. If it still continues moving around the circle, it has described an arc $= 4\pi$ when it reaches A for the second time. On reaching A for the third time the arc described is 6π, etc. Thus to each positive number in $\mathfrak{R}$ corresponds an arc; also, conversely, to each arc measured in the direction of the arrow corresponds a positive number in $\mathfrak{R}$.

With arcs measured in the negative direction are associated the negative numbers of $\mathfrak{R}$, and conversely.

202. From this mode of defining the circular functions we conclude at once the following properties:

The domain of definition of $\sin x$, $\cos x$ embraces all the numbers of $\mathfrak{R}$.

The domain of definition of $\tan x$ embraces all numbers of $\mathfrak{R}$ except

$$\frac{\pi}{2} + m\pi, \qquad (1$$

where $m = 0, \pm 1, \pm 2, \cdots$.

In fact, for these arcs, the secant OB is parallel to the tangent line AB, and therefore cuts off no segment on it. Thus for these values of the argument x, $\tan x$ is not defined.

Similarly, $\sec x$ is not defined for these same values 1); while $\operatorname{cosec} x$ is not defined for

$$x = m\pi. \qquad m = 0, \pm 1, \pm 2, \cdots \qquad (2$$

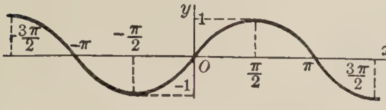
From similar triangles we have for all x, except these singular values in 1) or 2),

$$\tan x = \frac{\sin x}{\cos x}, \quad \sec x = \frac{1}{\cos x}, \quad \operatorname{cosec} x = \frac{1}{\sin x}.$$

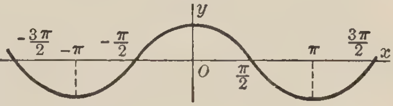
We observe that these relations involve division by 0, for the singular values 1) or 2).

From the above definition of *the circular functions* we see that they *are one-valued functions of* x.

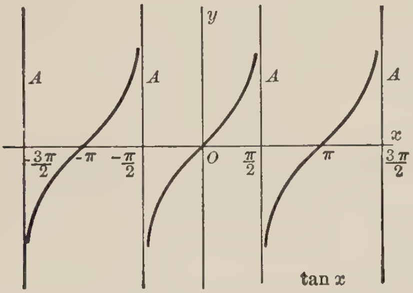
203. The graphs of the three principal functions $\sin x$, $\cos x$, $\tan x$, are given below.



sin x



cos x



tan x

204. The next most important property of the circular function is their *periodicity*.

In general we define thus:

Let ω be a constant $\neq 0$. Let $f(x)$ be a one-valued function whose domain of definition D is such that, if x is any point of D, so is

$$x + m\omega, \qquad m = \pm 1, \ \pm 2, \ \cdots$$

If

$$f(x+\omega) = f(x) \tag{1}$$

for every x in D, we say $f(x)$ is *periodic*, and *admits the period* ω.

If ω is a period of $f(x)$, so is $m\omega$.

$$m = \pm 1, \ \pm 2, \ \cdots$$

For,

$$f(x+2\omega) = f[(x+\omega)+\omega] = f(x+\omega) = f(x),$$

hence 2ω is a period. Similarly, 3ω, 4ω, $\cdots$ are periods.

On the other hand,

$$f(x-\omega) = f((x-\omega)+\omega) = f(x).$$

Hence $f(x)$ admits the period $-\omega$, and so -2ω, -3ω, etc.

If all the periods that $f(x)$ admits are multiples of a certain period $\tilde{\omega}$, this is the *primitive period* of $f(x)$, or *the period* of $f(x)$.

From trigonometry we have:

The period of $\sin x$, $\cos x$, *is* 2π; *the period of* $\tan x$ *is* π.

205. 1. *If* $f(x)$, $g(x)$ *admit the period* ω, *then*

$$f(x) \pm g(x), \tag{1}$$

$$f(x)g(x),\ \frac{f(x)}{g(x)},\ g(x) \neq 0, \tag{2}$$

admit also the period ω.

For example, let

$$h(x) = f(x) + g(x).$$

We have

$$h(x+\omega) = f(x+\omega) + g(x+\omega) = f(x) + g(x) = h(x).$$

2. *If the period of* $f(x)$ *is* ω, *the period of* $f(ax)$ *is* $\frac{\omega}{a}$. *Here* a *is any number* $\neq 0$.

For, let

$$g(x) = f(ax),$$

and let τ be any period of $g(x)$.

Then

$$g(x+\tau) = g(x),$$

or

$$f(a(x+\tau)) = f(ax).$$

This gives, setting

$$ax = t,$$

$$f(t + a\tau) = f(t).$$

Thus $a\tau$ is a period of $f(x)$; and therefore

$$a\tau = m\omega. \qquad m \text{ an integer.}$$

Hence

$$\tau = m\frac{\omega}{a}. \tag{3}$$

As $\frac{\omega}{a}$ is obviously a period of $g(x)$, it is *the* period of $g(x)$, by 3).

3. Suppose in 1, that ω, instead of being any period of f and g, is *the* period of these functions. It is important to note that we cannot infer that therefore ω is *the* period of the functions in 1), 2).

Ex. 1. Let

$$f(x) = \sin x, \quad g(x) = 4 \sin x \cos^2 x.$$

The period of these functions is 2π.

Yet the period of

$$h(x) = g(x) - f(x) = \sin 3x$$

is $\frac{2\pi}{3}$.

Ex. 2. Let

$$f(x) = \sin x, \quad g(x) = \cos x.$$

Then

$$h(x) = f(x)g(x) = \tfrac{1}{2} \sin 2x.$$

The period of f and g is 2π; the period of h is π.

Ex. 3. Let $f(x)$, $g(x)$, be as in Ex. 2. Let

$$h(x) = \frac{f(x)}{g(x)} = \tan x.$$

The period of h is again π.

Ex. 4. Let

$$f(x) = \sin^2 x, \quad g(x) = \cos^2 x.$$

The period of f, g is π.

But

$$h(x) = f(x) + g(x) = 1,$$

which has no *primitive* period.

206. From the periodicity of the circular functions we can prove that

The circular functions are transcendental.

Consider, for example,

$$y = \sin x.$$

If this is algebraic, let it satisfy the irreducible equation

$$y^n + R_1(x)y^{n-1} + \cdots + R_n(x) = 0. \qquad (1$$

Replace here x by $x + 2m\pi$, m an integer.

If we set

$$R_\kappa(x + 2m\pi) = T_\kappa(x),$$

1) gives, since y is unaltered,

$$y^n + T_1(x)y^{n-1} + \cdots + T_n(x) = 0. \qquad (2$$

As y satisfies both 1) and 2), it satisfies their difference

$$(R_1 - T_1)y^{n-1} + \cdots + (R_n - T_n) = 0.$$

Thus y satisfies an equation of degree $< n$, *which is not an identity.* As we assumed 1) is irreducible, this is a contradiction.

207. Another important property of the circular function is their *addition theorem*, which is expressed in the formulæ

$$\sin(x+y) = \sin x \cos y + \cos x \sin y,$$

$$\cos(x+y) = \cos x \cos y - \sin x \sin y,$$

$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y},$$

etc.

208. 1. Let $f(x)$ be a one-valued function whose domain of definition, D, is such that if x is any point of D, so is $-x$.

Let
$$f(-x) = f(x),$$

for every x in D. We say, then, that $f(x)$ is an *even function.*

If
$$f(-x) = -f(x),$$

we say $f(x)$ is an *odd function.*

Obviously,

The functions $\sin x$, $\tan x$ *are odd, while* $\cos x$ *is even.*

2. *Letting* O, O_1, O_2 *represent odd functions, and* E, E_1, E_2 *even functions, we have:*

$$O \pm O_1 = O_2, \quad E \pm E_1 = E_2,$$

$$O \cdot O_1 = E_2, \quad E \ E_1 = E_2, \quad O \cdot E = O_1,$$

$$\frac{O}{E} = O_1, \qquad \frac{E}{O} = O_1.$$

For example:

$$O_2(-x) = O(-x) \pm O_1(-x) = -O \mp O_1 = -(O \pm O_1) = -O_2.$$

The Exponential Functions

209. Let $a>0$ be a constant; the exponential functions are defined by

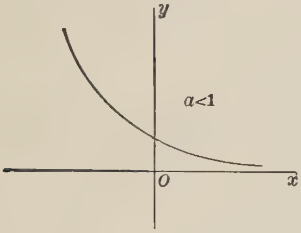
$$y=a^x.$$

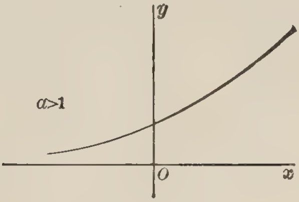
The domain of definition of y is $\Re$, and y is a one-valued function.

When $a=1$, the corresponding exponential function reduces to a constant, viz.:

$$y=1.$$

The graphs of y fall into two classes, according as $a \gtrless 1$.





An important exponential function is that corresponding to $a=e$.

$$e=2.71818\cdots$$

210. 1. The only properties of the exponential functions which we care to note now are the following:

The exponential function is nowhere equal to 0, *or any negative number.*

See 165, 2.

2. The *addition theorem* is expressed by

$$a^x a^y = a^{x+y}.$$

One-valued Inverse Functions

211. 1. The two remaining classes of functions, viz. the logarithmic and inverse circular functions, are inverse functions. Before considering them, we wish to develop the notion of *inverse functions* in general.

2. Let $f(x)$ be a one-valued function defined over a domain D.*

If
$$f(x'') > f(x')$$
for every pair of values $x'' > x'$ in D, we say $f(x)$ is an *increasing function* in D.

If on the contrary
$$f(x'') < f(x'), \qquad x'' > x',$$
we say $f(x)$ is a *decreasing function.*

If f is either an increasing or a decreasing function in D, but we do not care to specify which, we say it is *univariant.*

These definitions are extensions of those given in 108. The corresponding extension of the terms *monotone*, *monotone increasing*, *monotone decreasing* to function is obvious.

3. **Ex. 1.** For the domain $D = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, $\sin x$ is an increasing function. For the domain $D = \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$, $\sin x$ is a decreasing function.

Ex. 2. For the domain $D = \mathfrak{R}$, a^x is an increasing function if $a > 1$; it is a decreasing function if $a < 1$. Thus whether $a \gtrless 1$, a^x is a univariant function in $\mathfrak{R}$.

212. Let
$$y = f(x) \tag{1}$$
be a one-valued univariant function, defined over a domain D. Let E be the domain over which the variable y ranges.

We put the points of D and E in correspondence with each other as follows: two points x, y shall correspond to each other, or be associated, when they satisfy 1).

Then to a given x corresponds only *one* y, since $f(x)$ is one-valued. On the other hand, to a given y corresponds only *one* x, since $f(x)$ is univariant.

Thus to any x of D corresponds one, and only one, y of E; conversely, to any y of E corresponds one, and only one, x of D.

213. The considerations of the last article have led us to one of the most important notions of modern mathematics, that of *correspondence.*

---

* Such an expression as this will be constantly employed in the future. It does not mean that D includes all the values for which $f(x)$ may be defined, but only such values as one chooses to consider for the moment.

Let A and B be two sets of objects. Let us suppose that A and B stand in such a relation to each other, that to any object a of A correspond certain objects $b, b', b'', \cdots$ of B; and to any object b of B correspond certain objects $a, a', a'', \cdots$ of A.

Then A and B are said to be in correspondence.

If to each a corresponds only one b, and conversely, the correspondence is *one to one* (1 to 1), or *uniform.*

If to each a correspond m objects of B, and to each b correspond n objects of A, the correspondence is *m to n.*

In many cases, to each element of A correspond an infinity of objects of B, or conversely.

214. Let us return to 212. The correspondence we established between the points of D and E is uniform. This fact may be used to define a one-valued function $g(y)$, over the domain E. In fact, let x correspond to y. Then $g(y)$ shall have the value x, at the point y. Then

$$x = g(y).$$

The function g, just defined, is called the *inverse function of f.*

Evidently g is a one-valued function in E. It is also univariant.

In fact, to fix the ideas, suppose f is an increasing function.

Then, if

$$x' < x'',$$

we have

$$y' < y''.$$

Suppose now g were not an increasing function. Then for at least one pair of points,

$$y' < y'', \tag*{(1}$$

we would have

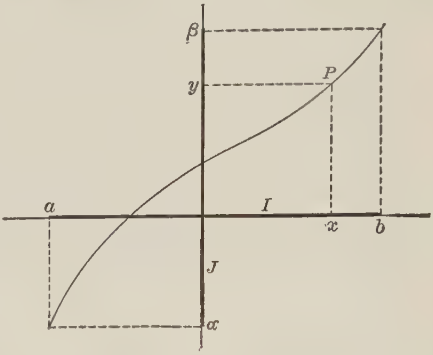
$$x' \overline{\gtrless} x''.$$

We cannot have $x' = x''$; for then $y' = y''$, which contradicts 1). We cannot have $x' > x''$; for then $y' > y''$, which again contradicts 1).

We have thus the theorem:

Let $y = f(x)$ be a one-valued univariant function, defined over a domain D. Let E be the domain of the variable y. Then the inverse function, $x = g(y)$, is one-valued and univariant in E.

215. The notion of inverse functions developed in 212 and 214 is quite general. It will perhaps assist the reader if we take a very simple case.



For the domain D let us take an interval $I=(a, b)$. For $f(x)$, let us take an increasing function, with graph as in the figure. The domain of y is then the interval $J=(\alpha, \beta)$. That the correspondence between the points of I and J, as defined in 212, is uniform, is seen here at once. For, to find the points y corresponding to a given x, we erect the ordinate at x. This cuts the graph but once, viz. at P. There is thus but one point y in J corresponding to the point x in I.

Similarly, to find the points x, corresponding to a given y, we draw the abscissa through y. This cuts the graph but once, viz. at P.

There is thus but one point x in I corresponding to a given y in J.

That the inverse function is one-valued, and is an increasing function in J, is at once evident from the figure.

The Logarithmic Functions

216. 1. We saw in 209 that the exponential functions

$$y=a^x, \quad a>0, \neq 1$$

are one-valued univariant functions for the domain $\mathfrak{R}$. The domain of the variable y is the interval $I=(0^*, +\infty)$. See 188.

Then, by 214, the inverse of the exponential functions are one-valued univariant functions, defined over I. By 174, these inverse functions are

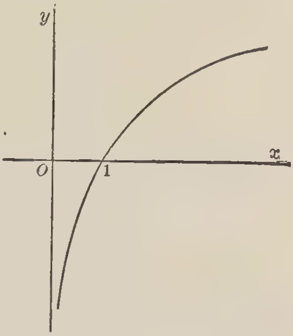
$$x=\log_a y, \tag{1}$$

and are called logarithmic functions with base a. In higher mathematics it is customary to take $a=e=2.71818\cdots$ When no ambiguity can arise, we may drop the subscript a in 1). Unless otherwise stated, we shall suppose the base is e.

2. The graph of the logarithmic function

$$y = \log x$$

is given in the figure.



3. The only other property of $\log x$ which we wish now to mention is their *addition theorem*,

$$\log xy = \log x + \log y.$$

Many-valued Inverse Functions

217. The circular functions give rise to many-valued inverse functions. It is easy to extend the considerations of 212 and 214 so as to arrive at the notion of *many-valued inverse functions* in all its generality.

Let

$$y = f(x) \qquad (1$$

be a one or many valued function, defined over a domain D. Let the domain of the variable y be E. We put the points of D and E in correspondence as follows: two points x, y shall correspond to each other or be associated when they satisfy 1). Then, to each y of E correspond one or more values of x, say

$$x, x', x'', \cdots \qquad (2$$

We define now a function $g(y)$ over E by assigning to g the values 2) of x associated with each point y of E.

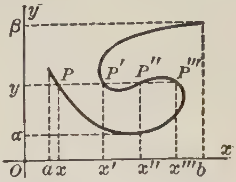
Then

$$x = g(y) \qquad (3$$

is the inverse function, defined by 1).

The equation 3) may be considered as the solution of 1) with respect to x.

218. To illustrate the rather abstract considerations of the last article, let us consider the following simple case, from a geometric standpoint.



Let the graph of

$$y = f(x)$$

be that in the figure.

Then $D = (a, b)$, and $E = (\alpha, \beta)$.

The greatest number of values of y for a given x in D is 3. Hence y is a three-valued function. Let y be a point of E. To find the points of D associated with it, we draw the abscissa through y. Let it cut the curve in the points $P, P', P'', \cdots$ The projections $x, x', x'', \cdots$ of these points P on the x-axis are the points sought.

The greatest number of values x corresponding to any y of E is 4. Hence the inverse function

$$x = g(y)$$

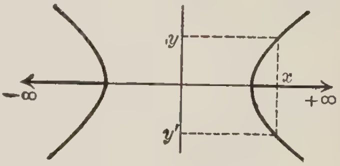
is a four-valued function.

219. Let us consider the function $y = f(x)$ defined by

$$x^2 - y^2 - 1 = 0, \qquad (1$$

or

$$y = \pm\sqrt{x^2 - 1}. \qquad (2$$



Its graph, given in the figure, is a hyperbola.

To a value of $x > 1$, or $x < -1$, correspond two values of y, marked y and y' in the figure. The domain D of x is marked heavy in the figure, and embraces all the points of the x-axis, except $(-1^*, 1^*)$. The domain E of y is the whole y-axis.

To any point y of E correspond two values of x, falling in D.

The inverse function $x = g(y)$, thus defined, is a solution of 1) or 2) with respect to x, viz.:

$$x = \pm\sqrt{1 + y^2}.$$

The correspondence which the equations 1) or 2) establish between the points of D and E is a 2 to 2 correspondence.

220. The preceding example illustrates the fact that

The inverse of an algebraic function which is not a constant is an algebraic function.

To prove this theorem, let $y=f(x)$ be defined by

$$P_0(x)y^n+P_1(x)y^{n-1}+\cdots+P_n(x)=0, \tag{1}$$

where the P's are polynomials in x, with constant coefficients.

The inverse function

$$x=g(y) \tag{2}$$

also satisfies 1). Let us arrange 1) with respect to x. If m is the highest degree of x in this equation, we get

$$Q_0(y)x^m+Q_1(y)x^{m-1}+\cdots+Q_m(y)=0. \tag{3}$$

As 2) satisfies 3), the inverse function 2) is an algebraic function also.

In this example, $y=f(x)$ is, *in general*, an n-valued function, while $x=g(y)$ is an m-valued function.

The correspondence that the equation 1) or 3) establishes between the points of D and E, the domains of the variables x, y, is thus an *n to m correspondence.*

The Inverse Circular Functions

221. These are the functions

$$\sin^{-1} x,\ \cos^{-1} x,\ \tan^{-1} x, \text{ etc.}$$

We prefer to follow continental usage, and denote them respectively by

$$\text{Arc}\sin x,\ \text{Arc}\cos x,\ \text{Arc}\,\text{tg}\, x, \text{ etc.}$$

We shall not take the space needed to treat all these functions; we take one of them, Arc sin, as an illustration. The others may be treated in the same way.

We start with the equation

$$y=\sin x, \tag{1}$$

whose graph is given in 203.

The domain D, over which $\sin x$ is defined, is $\mathfrak{R}$; the domain of y is $E = (-1, 1)$.

Let y be a point of E. If x_0 is one of the associated points of D, all the points of D associated with y are given by

$$\left.\begin{aligned} &x_0 + 2m\pi, & &(2\\ &\pi - x_0 + 2m\pi, & &(3 \end{aligned}\right\} \quad m = 0, \ \pm 1, \ \pm 2 \cdots$$

as is shown in trigonometry.

Thus, to a given value of y there are a double infinity of values of x.

The inverse function defined by 1), viz.:

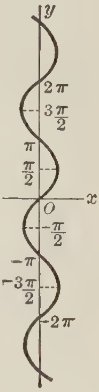
$$x = \text{Arc} \sin y,$$

has the interval $E = (-1, 1)$ for its domain of definition. It is an infinite-valued function whose values for a given y are given in 2), 3).

222. The graph of

$$y = \text{Arc} \sin x$$

is given in the adjoining figure.



The reader will observe that this graph can be got at once from the graph of $\sin x$ (see 203) by turning it around and changing the axes.

This property is obviously true of the graph of any inverse function.

Thus, if the graphs of

$$e^x, \ \cos x, \ \tan x, \ \text{etc.},$$

are given, we may get at once the graphs of

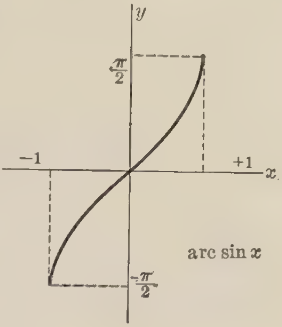
$$\log x, \ \text{Arc} \cos x, \ \text{Arc} \operatorname{tg} x, \ \text{etc.}$$

223. The treatment of many-valued functions is much simplified by employing the notion of a *branch* of the function. This will be explained when we have considered the notion of continuity. For the present, however, we wish to define what are called the *principal branches of the inverse circular functions.*

Looking at the graph of Arc sin x given in 222, we see we can define a *one-valued function* over the interval $(-1, 1)$ by taking those values of Arc sin x which fall in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

The function so defined is called the *principal branch of the Arcsin function.* We shall denote it by arc sin x.

Its graph is given in the adjoining figure.



224. 1. The *principal branch of Arc cos x* is formed of those values of this function which fall in the interval $(0, \pi)$.

The one-valued function so defined over the interval $(-1, 1)$ is denoted by arc cos x.

Its graph is given in Fig. 1.

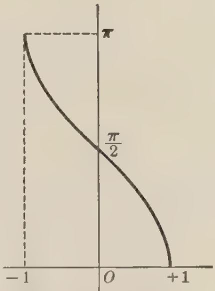


FIG. 1. arc cos x

2. The *principal branch of Arc tg x* is formed of those values of this function which fall in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

The one-valued function so defined over $(-\infty, \infty)$ is denoted by arc tg x.

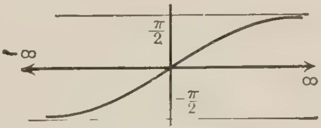


FIG. 2. arc tg x

Its graph is given in Fig. 2.

FUNCTIONS OF SEVERAL VARIABLES

The Rational and Algebraic Functions

225. In the list of the elementary functions given in 186, the first three, viz. the integral rational, the rational, and the algebraic functions, are, in general, functions of several variables.

For simplicity, we treated them first as functions of a single variable. We wish now to define them in all their generality. At the same time we shall consider the general notion of functions of several variables and certain related geometric ideas.

226. 1. An *integral rational function of n variables* $x_1, x_2 \cdots x_n$ is an expression of the type

$$y = Ax_1^{m_1}x_2^{m_2} \cdots x_n^{m_n} + Bx_1^{l_1}x_2^{l_2} \cdots x_n^{l_n} + \cdots + Lx_1^{e_1}x_2^{e_2} \cdots x_n^{e_n}. \quad (1$$

Here $A, B, \cdots L$ are constants, and the exponents $m, l, \cdots e$ are positive integers or 0. Such functions are

$$2\,x_1^2x_3 + x_2 + 5\,x_1x_2x_3, \quad (2$$

$$ax_1x_2^2x_3 + bx_2^5x_3 + cx_3^5 + dx_1^7x_2^5x_3. \quad (3$$

We may write 1) in the form

$$y = \Sigma A_{m_1m_2\cdots m_n}x_1^{m_1}x_2^{m_2} \cdots x_n^{m_n} \quad (4$$

where the summation extends over all the terms of y.

A still shorter notation is

$$y = \Sigma Ax_1^{m_1}x_2^{m_2} \cdots x_n^{m_n} \quad (5$$

which may be employed when no ambiguity can arise.

The greatest of all the sums of the exponents

$$m_1 + m_2 + \cdots + m_n, \quad l_1 + l_2 + \cdots + l_n \cdots$$

is the *degree* of y.

Thus the degree of 2) is 3; the degree of 3) is 13.

2. When the degree of each term of 1) is the same, it is said to be *homogeneous*.

Let

$$F = \Sigma Ax_1^{m_1}x_2^{m_2} \cdots x_n^{m_n}$$

be homogeneous and of degree m. If in F we replace x_1 by $\lambda x_1, \cdots x_n$ by λx_n, and denote the result by $\overline{F}$, we have

$$\overline{F} = \lambda^m F.$$

For,

$$\overline{F} = \Sigma A\lambda^{m_1+\cdots+m_n}x_1^{m_1} \cdots x_n^{m_n}.$$

But, for all the terms of F,

$$m_1 + \cdots + m_n = m.$$

Hence

$$\overline{F} = \lambda^m \Sigma A x_1^{m_1} \cdots x_n^{m_n} = \lambda^m F.$$

3. When y is of degree 1, we have

$$y = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n + a_0.$$

It is said to be a *linear integral function* of the x's. If $a_0 = 0$, it becomes

$$y = a_1 x_1 + \cdots + a_n x_n,$$

which is the general type of a *linear homogeneous integral function* of the x's.

In algebra, integral rational functions are called *polynomials*.

227. 1. To get a value of

$$y = \Sigma A x_1^{m_1} \cdots x_n^{m_n} \quad (1$$

we give to each of the variables x a certain numerical value, as,

$$x_1 = a_1, \; x_2 = a_2, \; \cdots \; x_n = a_n. \quad (2$$

These values put in 1) give the corresponding value of y, say $y = b$.

When $n = 1, 2, 3$, we can represent geometrically the values 2) by a point on a right line, a point in a plane, or a point in space, respectively, viz. the point a whose coördinates are a_1, or a_1, a_2, or a_1, a_2, a_3. If we give the x's different sets of values, we get different points in 1, 2, or 3 dimensional space. As in the case of one variable, we can say y has the value b at the point a.

2. It is convenient to extend these and other geometric terms, employed when the number of variables $n = 1, 2, 3$, to the case when $n > 3$. Thus any complex of n numbers, $a_1, a_2 \cdots a_n$, is called a *point;* $a_1, a_2, \cdots$ are called its *coördinates*. We denote the point by

$$a = (a_1, a_2, \cdots a_n).$$

The aggregate of all possible points, the x's running over all the numbers in $\mathfrak{R}$, we call an *n-dimensional space* or an *n way space;* and denote it by $\mathfrak{R}_n$.* Later we shall extend the terms *distance*, *sphere*, *cube*, etc., to $\mathfrak{R}_n$. Cf. 244. The reader is not to suppose for a moment that there really is an n-dimensional space, or an n-dimensional cube, in the ordinary empirical sense of the word; but to bear in mind that these terms are merely names for certain numerical aggregates.

3. Employing this geometrical language, we may say that,

The integral rational function of several variables, say n variables, is a one-valued function whose domain of definition embraces all the points of $\mathfrak{R}_n$.

228. As in the case of one variable, the *rational function of several variables* is the quotient of two integral rational functions in these variables. Its general expression is, therefore,

$$R = \frac{\Sigma A x_1^{r_1} \cdots x_n^{r_n}}{\Sigma B x_1^{s_1} \cdots x_n^{s_n}} = \frac{F}{G}. \qquad (1$$

Its domain of definition embraces all the points of $\mathfrak{R}_n$, except those points at which G vanishes, which we call poles of R. For all points of this domain, R is a one-valued function.

If m' is the degree of F, and m'' is that of G, the *degree of R* is the greater of the two integers m', m''.

When the degree of R is 1, it is called a *linear rational function.* Its general expression is

$$\frac{a_0 + a_1 x_1 + \cdots + a_n x_n}{b_0 + b_1 x_1 + \cdots + b_n x_n}. \qquad (2$$

We say R is *homogeneous* when F and G are homogeneous. We have evidently, as in 226, 2,

$$R(\lambda x_1, \lambda x_2, \cdots \lambda x_n) = \lambda^t R(x_1 \cdots x_n), \qquad (3$$

where t is an integer, positive, negative, or zero.

229. The definition of an algebraic function of n variables is an obvious extension of that given for one variable, in 197. Thus y is an algebraic function of $x_1, x_2, \cdots x_n$, when it satisfies an equation of the type

$$y^n + R_1 y^{n-1} + \cdots + R_{n-1} y + R_n = 0, \qquad (1$$

where the coefficients R are rational functions of $x_1 \cdots x_n$, and n is a positive integer.

For any point $x = a$ in $\mathfrak{R}_n$, for which none of the denominators of the R's vanish, y has at most n values.

Thus y is at most an n-valued function. Its domain of definition embraces all points of $\mathfrak{R}_n$ except the poles of the coefficients R, and those points for which 1) has no real root.

Functions of Several Variables in General

230. We can give now the definition of a function in n variables. Let $x = (x_1, x_2, \cdots x_n)$ range over the points of a certain domain D, viz. over $\mathfrak{R}_n$ or a part of it. Let a law be given which assigns to y one or more values for each point of D. We say y is a function of $x_1, x_2, \cdots x_n$, and write

$$y = f(x_1 \cdots x_n), \text{ or } y = \phi(x_1 \cdots x_n), \text{ etc.}$$

When no ambiguity can arise, we may even write

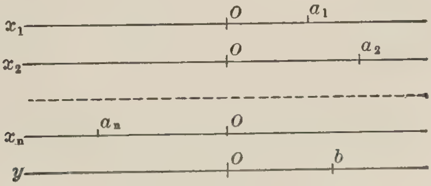
$$y = f(x), \quad y = \phi(x), \text{ etc.,}$$

where x stands for the n variables $x_1 \cdots x_n$.

The meaning of the terms of 189, viz. *one-valued* and *many-valued*, *independent variables* or *argument*, *dependent variable*, *domain of definition of the function*, when applied to several variables, needs no explanation.

231. We explain now the graphical representation of functions of several variables.

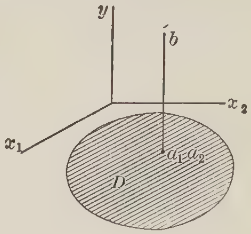
We take $n + 1$ axes, one for each of the variables $y, x_1, x_2, \cdots x_n$.



The representation of a point $x_1 = a_1 \cdots x_n = a_n$, is a complex of n points $a_1 \cdots a_n$, as in the figure. The value of y, say $y = b$, is represented by the point b on the y-axis. This representation, although unsatisfactory in some respects, is still often useful.

232. When $n=2$, we have two other modes of representation. Let the function be

$$y=f(x_1, x_2).$$

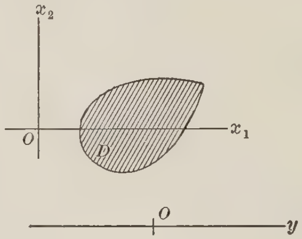


We take three axes, x_1, x_2, y, as in analytic geometry, of three dimensions. To the set of values of the independent variables

$$x_1=a_1,\ x_2=a_2$$

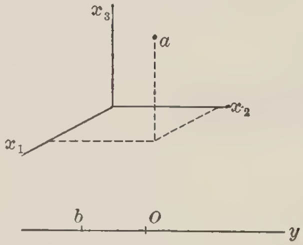
corresponds the point $a=(a_1, a_2)$, whose coördinates are a_1, a_2. The value b of y at this point we lay off on the ordinate through a. As x runs over its domain D, y will ordinarily trace out a surface in $\mathfrak{R}_3$.

233. The other mode of representation is by means of a plane and an axis.



The domain of the independent variables we represent by points in the x_1x_2 plane, while y is represented by points laid off on a separate axis, as in the figure.

234. When $n=3$, we may employ the following representation.



Let

$$y=f(x_1x_2x_3)$$

be defined over a domain D.

To represent D, we take three rectangular axes.

To the set of values

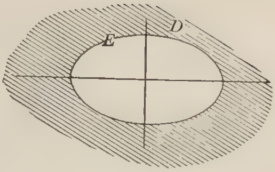
$$x_1=a_1, \quad x_2=a_2, \quad x_3=a_3$$

corresponds the point a, whose coördinates are a_1, a_2, a_3. The values of y we lay off on a separate axis, as in the figure.

235. From the elementary functions of one variable we can build an infinity of functions of several variables. We give some examples which *illustrate the various domains of definition* that a function of several variables may have. We shall take $n=2$.

Ex. **1.**

$$z = \log\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right).$$



For points within the ellipse E, whose equation is

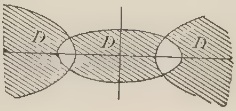
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0,$$

the argument of z is negative. For points on E the argument is 0. As the logarithmic function is defined only for positive values of the argument, the domain of definition D, of z, is the region shaded in the figure. Its edge, or E, does not belong to D.

236. Ex. 2.

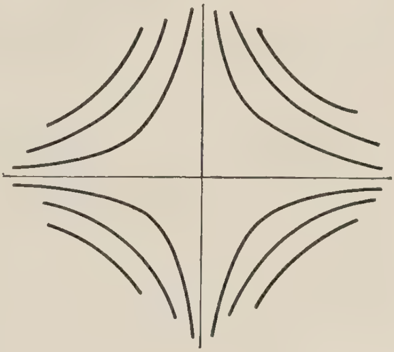
$$z = \log\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)\left(\frac{x^2}{\alpha^2} - \frac{y^2}{\beta^2} - 1\right) = \log uv.$$

Since $\log uv$ is not defined, unless $uv > 0$, u and v must be both positive, or both negative. The domain of definition D, of z, is thus the region shaded in the figure. Since $uv = 0$ on the edge of D, these points do not belong to D.



237. Ex. 3.

$$z = \tan \tfrac{1}{2}\pi xy.$$



Since $\tan u$ is not defined when

$$u = \frac{\pi}{2} + m\pi,$$

$$m = 0, \pm 1, \pm 2, \cdots$$

we see the domain of definition of z includes all the points of the xy plane, except a family of hyperbolas

$$xy = 2m + 1.$$

Composite Functions

238. 1. An extremely useful notion in many investigations is that of a *function of functions*, or *composite functions*. Let

$$u_1 = f_1(x_1 \cdots x_n) \quad \cdot\cdot \quad u_m = f_m(x_1 \cdots x_n)$$

be defined over a domain X in n-dimensional space $\mathfrak{R}_n$. Let

$$u = (u_1 u_2 \cdots u_m)$$

be a point in an m-dimensional space $\mathfrak{R}_m$.

While x runs over X, let u run over a domain U. Let

$$y = \phi(u_1 \cdots u_m) \qquad (1$$

be defined over U. Then y is defined for every point x in X. We may, therefore, consider y as a function of the x's through the u's. We say y is a *function of functions*, or a *composite function.*

2. When speaking of composite functions, we shall always suppose, even without further mention, that the domain of definition of 1) is at least as great as U.

3. When x ranges over X, u, as we said, runs over the domain U. It is convenient for brevity to call U the *image* of X.

EXAMPLE.

$$u_1 = x_1 x_2, \quad u_2 = \sec x_1, \quad u_3 = e^{\frac{x_2}{x_1}}.$$

$$y = \log u_1 + \tan \frac{u_2}{u_1}.$$

Here u_1, u_2, u_3 are defined for all the points of $\mathfrak{R}_2$, for which

$$x_1 \neq 0 \text{ or } \frac{\pi}{2} + m\pi, \qquad m = 0, \ \pm 1, \ \pm 2, \ \cdots$$

while y is defined for all the points of $\mathfrak{R}_3$, for which

$$u_1 \neq 0, \text{ and } \frac{u_2}{u_1} \neq \frac{\pi}{2} + n\pi. \qquad n = \pm 1, \ \pm 2, \ \cdots$$

239. The notion of a composite function is sometimes useful in transforming a function as follows. Let

$$y = F(x_1 \cdots x_n).$$

The variable x may enter F in certain combinations, so that if we set

$$u_1 = \phi_1(x_1 \cdots x_n) \quad \cdots \quad u_m = \phi_m(x_1 \cdots x_n)$$

y goes over into

$$y = G(u_1 \cdots u_m).$$

EXAMPLE.

$$y = a\frac{x_1^2}{x_2^2} + \frac{x_1^2 + x_2^2}{x_1^2 - x_2^2} + \log \frac{x_1}{x_2} = F(x_1, x_2).$$

Let

$$u = \frac{x_1}{x_2},$$

then

$$y = au^2 - \frac{1 + u^2}{1 - u^2} + \log u = G(u).$$

Limited Functions

240. Let $f(x_1 \cdots x_m)$ be defined over a domain D. If there exists a positive number M, such that

$$|f| \leq M,$$

for every point of D, f is said to be *finite* or *limited* in D; otherwise f is *unlimited* in D.

Ex. 1.

$$f(x) = \sin x.$$

Since

$$|\sin x| \leq 1, \qquad x \text{ arbitrary},$$

$\sin x$ is a limited function for any domain.

Ex. 2.

$$f(x) = a_0 + a_1 x + \cdots + a_n x^n, \qquad a_n \neq 0,$$

is *limited* in any domain $(-G, G)$, where G is some fixed positive number.

It is unlimited in the domain $(0, +\infty)$, for example.

Ex. 3.

$$f(x) = \frac{1}{x}$$

is defined for every $x \neq 0$.

It is limited in any domain as (a, ∞), if $a < 0$.

It is unlimited in $(0^*, 1)$, for example.

241. *Let $f(x_1 \cdots x_n)$, $g(x_1 \cdots x_n)$ be limited functions in a domain D. Then*

$$f \pm g, \qquad fg$$

are limited in D.

If

$$|g| > a > 0$$

in D, then

$$\frac{f}{g}$$

is limited in D.

Since f, g are limited in D, let

$$|f|, |g| < M.$$

Then

$$|f \pm g| \leq |f| + |g| < 2M.$$

Hence $f \pm g$ is limited in D.

Also

$$|fg| = |f| \cdot |g| < M^2.$$

Hence fg is limited in D.

Finally,

$$\left|\frac{f}{g}\right| = \frac{|f|}{|g|} < \frac{M}{a}.$$

Hence $\frac{f}{g}$ is limited in D.

CHAPTER V

FIRST NOTIONS CONCERNING POINT AGGREGATES

Preliminary Definitions

242. In elementary mathematics, the functions employed are usually defined by simple analytic expressions. Their nature is simple, and their domains of definition receive little attention. In the theory of functions we take a higher standpoint, and consider functions defined by any law, as explained in 189 and 230. Such functions are not tied down to an analytic expression; indeed, we may not know how to form their analytic expressions.

From this point of view, the domain D over which the function is defined or spread out is often of great importance. Frequently we choose first the domain D, and then define a function for the points of D.

The domain of definition of a function of n variables may be *any* set or aggregate of points in $\mathfrak{R}_n$. We wish to treat now the most elementary properties of such aggregates which we call *point aggregates.*

243. 1. Two point aggregates A, B are *equal*, when every point of A lies in B, and every point of B lies in A. In this case, we write

$$A = B.$$

2. If every point of B lies in A but not every point of A lies in B, we say B is a *partial* or *sub aggregate* of A, and write

$$A > B \text{ or } B < A.$$

3. If A does not exist, *i.e.* if it contains no points, we write

$$A = 0.$$

As the symbol 0 also stands for the origin, we shall write, in case of ambiguity,

$$A = (0),$$

when we wish to indicate that A consists of the origin alone.

The fact that A contains at least one point, we indicate by

$$A > 0.$$

4. Let A, B be two point aggregates having no point in common. The aggregate formed by their reunion is called their *sum*, and is denoted by

$$A + B.$$

5. If B is a partial aggregate of A, the aggregate formed by removing all the points of B from A is called the *difference of* A, B, and is denoted by

$$A - B.$$

It is also called the *complement of* B.

6. If a or x, for example, are general symbols for the points of an aggregate, we can represent the aggregate by

$$\{a\} \text{ or } \{x\}.$$

Thus, if $A = \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \cdots$

we can write $A = \left\{\frac{1}{n}\right\}.$

Or if $A = a_1, a_2, a_3, \cdots$

we can write $A = \{a_n\}.$

244. *Definitions of configurations in n-way space.* Cf. 227, 2.

1. Let $a = (a_1 \cdots a_n)$, $b = (b_1 \cdots b_n)$ be two points of $\mathfrak{R}_n$. We say

$$\sqrt{(a_1 - b_1)^2 + \cdots + (a_n - b_n)^2} \qquad (1$$

is the *distance* between a, b; we denote it by

$$\text{Dist}\,(a, b) \text{ or } \overline{a, b}.$$

2. The points x satisfying

$$x_1 - a_1 = \lambda(b_1 - a_1) \quad \cdots \quad x_n - a_n = \lambda(b_n - a_n) \qquad (2$$

lie on a *right line* L, viz. the line determined by the two points a, b. Here λ runs over all the numbers of $\mathfrak{R}$.

When $\lambda = 0$, $x = a$; when $\lambda = 1$, $x = b$. Points x, for which $0 \leq \lambda \leq 1$, form a *segment* or *interval* (a, b) of L. Such points are said to lie *between* a, b.

An aggregate lying on a right line is called *rectilinear*.

3. If three points a, b, c lie on a right line, we have from 2) that

$$\frac{c_\iota - a_\iota}{b_\iota - a_\iota} = \frac{c_\kappa - a_\kappa}{b_\kappa - a_\kappa}; \qquad \iota, \kappa = 1, 2, \cdots n. \tag{3}$$

and conversely, if 3) holds, a, b, c lie on a right line.

4. Let a, b be two points on the line L, and

$$r = \text{Dist}\,(a, b).$$

Then

$$\lambda_1 = \frac{a_1 - b_1}{r} \quad \cdots \quad \lambda_n = \frac{a_n - b_n}{r}$$

are the *direction cosines* of the line L. Obviously,

$$\lambda_1^2 + \cdots + \lambda_n^2 = 1.$$

5. The points x defined by

$$(x_1 - a_1)^2 + \cdots + (x_n - a_n)^2 = r^2, \qquad r > 0.$$

lie *on* a *sphere* S whose center is a and whose radius is r.

The equation of S may also be written

$$\text{Dist}\,(a, x) = r.$$

The points x, such that

$$\text{Dist}\,(a, x) < r$$

lie *within* S. If

$$\text{Dist}\,(a, x) > r,$$

x lies *without* S. If x lies *on* or *within* S, it lies *in* S.

6. The points x, such that

$$|x_1 - a_1| \leq \tfrac{1}{2} e \quad \cdots \quad |x_n - a_n| \leq \tfrac{1}{2} e,$$

form a *cube*, with center a and edge e.

7. The points x, such that

$$a_1 \leq x_1 \leq b_1 \quad \cdots \quad a_n \leq x_n \leq b_n,$$

form a rectangular *parallelopiped* or *cell* whose edges are of length

$$e_1 = b_1 - a_1 \quad \cdots \quad e_n = b_n - a_n.$$

8. The cube

$$|x_1 - a_1| \leq \frac{r}{\sqrt{n}} \quad \cdots \quad |x_n - a_n| \leq \frac{r}{\sqrt{n}}$$

is called the *inscribed cube* C of the sphere S, of radius r, and center a.

Every point x of C is in S. For,

$$\sqrt{(x_1 - a_1)^2 + \cdots + (x_n - a_n)^2} \leq \sqrt{\frac{r^2}{n} + \cdots + \frac{r^2}{n}} = r.$$

9. Let the cube C be given by

$$|x_1 - a_1| \leqq \tfrac{1}{2}\sigma \quad \cdots \quad |x_n - a_n| \leqq \tfrac{1}{2}\sigma.$$

The points

$$v_1 = a_1 \pm \tfrac{1}{2}\sigma \quad \cdots \quad v_n = a_n \pm \tfrac{1}{2}\sigma$$

are called the *vertices*.

If

$$v = (v_1 \cdots v_n)$$

is one vertex,

$$v' = (-v_1 + 2a_1, \ \cdots - v_n + 2a_n)$$

is called the *opposite vertex*.

The line joining a pair of opposite vertices evidently passes through the center of C. It is called a *diagonal*.

The length of a diagonal is

$$\sqrt{\sigma^2 + \cdots + \sigma^2} = \sigma\sqrt{n}.$$

10. The distance between two points a, b in C is greatest when they are opposite vertices. For, each term $(a_\iota - b_\iota)^2$ in 1) has then its greatest value, viz. σ^2.

11. If $e_1, e_2, \cdots e_m$ are the lengths of the edges of the parallelopiped in 6, we say the product

$$e_1 \cdot e_2 \cdots e_n$$

is its *volume*.

In case the parallelopiped is a cube of edge σ, its volume is σ^n.

12. The points x defined by

$$a_1x_1 + \cdots + a_nx_n + d = 0 \qquad (4$$

lie in a *plane*. The two planes 4) and

$$a_1x_1 + \cdots + a_nx_n + e = 0$$

are *parallel*.

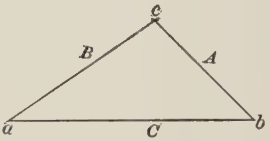
245. Let a, b, c be three points in $\mathfrak{R}_n$.

Let

$$A = \text{Dist}\,(b, c), \quad B = \text{Dist}\,(a, c), \quad C = \text{Dist}\,(b, a).$$

When $n = 1, 2, 3$, we have

$$A \leqq B + C. \qquad (1$$



Here the inequality sign holds unless A, B, C lie on a right line L.

We show now that 1) holds for every n.*

To this end, set

$$\alpha_\iota = b_\iota - c_\iota, \quad \beta_\iota = a_\iota - c_\iota, \quad \gamma_\iota = b_\iota - a_\iota,$$

where a_ι, b, c_ι; $\iota = 1, 2, \cdots n$, are the coördinates of a, b, c. Then

$$\alpha_\iota = \beta_\iota + \gamma_\iota. \qquad (2$$

Now,

$$A^2 = \alpha_1^2 + \cdots + \alpha_n^2 = \Sigma\alpha_\iota^2;$$

$$B^2 = \beta_1^2 + \cdots + \beta_n^2 = \Sigma\beta_\iota^2; \qquad (3$$

$$C^2 = \gamma_1^2 + \cdots + \gamma_n^2 = \Sigma\gamma_\iota^2. \qquad (4$$

From 2), we have also

$$A^2 = \Sigma(\beta_\iota + \gamma_\iota)^2 = \Sigma\beta_\iota^2 + \Sigma\gamma_\iota^2 + 2\,\Sigma\beta_\iota\gamma_\iota. \qquad (5$$

Thus to prove 1), we have to show that

$$A^2 \leq B^2 + C^2 + 2\,BC,$$

or, using 3), 4), 5), that

$$\Sigma\beta_\iota^2 + \Sigma\gamma_\iota^2 + 2\,\Sigma\beta_\iota\gamma_\iota \leq \Sigma\beta_\iota^2 + \Sigma\gamma_\iota^2 + 2\sqrt{(\beta_1^2 + \cdots + \beta_n^2)(\gamma_1^2 + \cdots + \gamma_n^2)}.$$

---

* If the reader finds the demonstration difficult, let him go through it, taking $n = 2$ or 3. We recommend this in the case of any demonstration which the reader finds too hard for general n.

This shows that it will suffice to prove that

$$(\Sigma\beta_\iota\gamma_\iota)^2 \leq \Sigma\beta_\iota^2\Sigma\gamma_\kappa^2. \tag*{(6}$$

To do this, we start from the inequality

$$(\beta_\iota\gamma_\kappa - \beta_\kappa\gamma_\iota)^2 \overset{-}{>} 0. \tag*{(7}$$

By 244, 3, the inequality sign holds for at least one pair of indices ι, κ, unless a, b, c lie on a right line.

From 7), we have

$$\beta_\iota^2\gamma_\kappa^2 + \beta_\kappa^2\gamma_\iota^2 \overset{=}{>} 2\,\beta_\iota\beta_\kappa\gamma_\iota\gamma_\kappa. \tag*{(8}$$

Let us form all the relations of this type, letting ι, κ run over the indices 1, 2, $\cdots$, n, and keeping $\iota \neq \kappa$.

If we add these, we get

$$\Sigma\beta_\iota^2\gamma_\kappa^2 \overset{=}{>} 2\,\Sigma\beta_\iota\beta_\kappa\gamma_\iota\gamma_\kappa. \qquad \iota \neq \kappa. \tag*{(9}$$

On the other hand,

$$\Sigma\beta_\iota^2 \cdot \Sigma\gamma_\kappa^2 = (\beta_1^2 + \cdots + \beta_n^2)(\gamma_1^2 + \cdots + \gamma_n^2) = \Sigma\beta_\iota^2\gamma_\iota^2 + \sum_{\iota\neq\kappa}\beta_\iota^2\gamma_\kappa^2.$$

Hence, by 9),

$$\Sigma\beta_\iota^2 \cdot \Sigma\gamma_\kappa^2 \overset{=}{>} \Sigma\beta_\iota^2\gamma_\iota^2 + 2\sum_{\iota\neq\kappa}\beta_\iota\beta_\kappa\gamma_\iota\gamma_\kappa. \tag*{(10}$$

But

$$(\Sigma\beta_\iota\gamma_\iota)^2 = (\beta_1\gamma_1 + \cdots + \beta_n\gamma_n)^2 = \Sigma\beta_\iota^2\gamma_\iota^2 + 2\sum_{\iota\neq\kappa}\beta_\iota\beta_\kappa\gamma_\iota\gamma_\kappa.$$

This in 10) gives 6).

246. A point x for which Dist (a, x) is small, is said to be *near* a. What is to be considered as *small*, depends on the problem in hand.

The points x, such that

$$\text{Dist}\,(a, x) \leq \rho, \qquad \rho > 0.$$

form an aggregate called the *domain of the point* a, *of norm* ρ. It is denoted by

$$D_\rho(a), \qquad D(a), \qquad D_\rho.$$

For example, in $\mathfrak{R}_1$, $D_\rho(a)$ is the interval $(a-\rho, a+\rho)$.

In $\mathfrak{R}_2$, $D_\rho(a)$ embraces all points *in* a circle of radius ρ, and center a; in $\mathfrak{R}_3$, it embraces all points in a sphere of radius ρ.

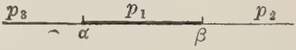
We sometimes wish to exclude the point a from its domain. When this is done, the domain is said to be *deleted;* we denote it by

$$D^*_\rho(a) \quad \text{or} \quad D^*(a).$$

247. Let A be a point aggregate in $\mathfrak{R}_n$.

Let p be any point in $\mathfrak{R}_n$. We say p is an *inner point* of A if every point in some domain of p lies in A, *i.e.* if there exists a $\rho > 0$ such that every point of $D_\rho(p)$ lies in A. The point p is an *outer point* of A if no point of $D_\rho(p)$ lies in A, however small $\rho > 0$ is taken. Finally, p is a *frontier point* of A if in every $D_\rho(p)$, however small $\rho > 0$ is taken, there is at least one point of A and one point not in A. Every point of $\mathfrak{R}_n$ is either an inner, an outer, or a frontier point of A. The frontier points of a cube or parallelopiped form its *surface.*

248. Ex. 1. $A = (\alpha, \beta)$.



Here any point p_1, such that $\alpha < p_1 < \beta$, is an inner point. Any point p_2, such that $p_2 > \beta$ or $p_3 < \alpha$, is an outer point.

The frontier points are α and β.

Ex. 2. A embraces the rational points in (α, β). Here all points p, such that $p < \alpha$ or $p > \beta$, are outer points. The points of A are *all* frontier points. For, if a be any point of A, there are irrational points in every $D_\rho(a)$, however small $\rho > 0$ is taken, by 84.

In this example A contains no inner points.

Ex. 3. A embraces all the points in $\mathfrak{R}_2$, both of whose coördinates are rational.

Here every point p of $\mathfrak{R}_2$ is a frontier point. In fact, consider a little circle C of radius $\rho > 0$ and center p. Evidently p contains points in A and points not in A, however small ρ is taken.

In this example there are no outer and no inner points of A.

249. *Let b be an inner point of*

$$S = D_\sigma(a).$$

Then

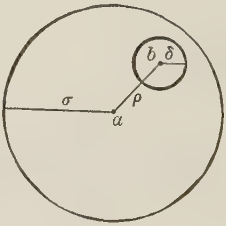
$$\Delta = D_\delta(b)$$

lies within S if

$$\rho + \delta < \sigma, \qquad (1$$

where

$$\rho = \text{Dist}\,(a, b).$$



The theorem is proved if we show that the points y of Δ satisfy the relation

$$\mathrm{Dist}\,(a, y) < \sigma. \tag{2}$$

But, by 245,

$$\mathrm{Dist}\,(a, y) \leq \mathrm{Dist}\,(a, b) + \mathrm{Dist}\,(b, y) = \rho + \delta.$$

Thus, by 1), the relation 2) is valid.

250. 1. Let A be a point aggregate, and p any point in $\mathfrak{R}_n$. The points of A, lying in $D_\rho(p)$, form *the vicinity of p, of norm ρ*. It is denoted by

$$V_\rho(p) \quad \text{or} \quad V(p).$$

Thus $D(p)$ embraces *all* points near p, while $V(p)$ includes *only* points of A, near p.

EXAMPLE. Let $A = 1, \frac{1}{2}, \frac{1}{3}, \cdots$

Here $D_\rho(0)$ is the *interval* $(-\rho, \rho)$, while $V_\rho(0)$ is the set of points

$$\frac{1}{m}, \frac{1}{m+1}, \frac{1}{m+2}, \cdots$$

where m is the least integer such that $\frac{1}{m} \overline{\gtrless} \rho$.

The point p may or may not lie in $V(p)$. We sometimes wish expressly to exclude it. When this is done, the resulting aggregate is the *deleted vicinity of p*; it is denoted by

$$V_\rho^*(p) \quad \text{or} \quad V^*(p).$$

251. When treating functions of a single variable x, we have often to consider the behavior of the function on one side of a point a. This leads us to split the domain and vicinity of a into two parts, forming a *right* and *left hand domain*; a *right* and *left hand vicinity* of a.

The right hand domain and vicinity we denote respectively by

$$RD(a), \quad RV(a).$$

The left hand domain and vicinity are denoted by

$$LD(a), \quad LV(a).$$

The point a lies in both the right and left hand domain. It lies in both the right and left hand vicinity if a lies in $V(a)$. It should be remembered that these terms refer only to rectilinear aggregates.

252. 1. A point aggregate is said to be *finite* when it contains only a finite number of points. Otherwise it is *infinite.*

A point aggregate A is said to be *limited* when all its points lie within a certain sphere or cube, having the origin as center.

This definition is equivalent to saying that the coördinates a_1, $a_2, \cdots a_n$, of every point of A, are numerically less than some positive number M. If A is not limited, it is said to be *unlimited.* Obviously: *Every finite aggregate is limited.*

Ex. 1. $$A = 1,\ 2,\ 3,\ \cdots$$
is an infinite unlimited aggregate.

Ex. 2. $$A = 1,\ \tfrac{1}{2},\ \tfrac{1}{3},\ \cdots$$
is an infinite limited aggregate.

Ex. 3. $$A = \text{points of the interval } (\alpha, \beta)$$
is an infinite limited aggregate.

2. In the case of a rectilinear aggregate A, it may happen that the coördinates of all its points x are less than some number M. We say A is *limited to the right.*

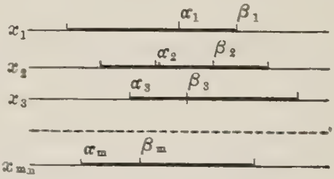
If the coördinates of all the points x are greater than some number N, we say A is *limited to the left.*

Ex. 1. $$A = 10,\ 9,\ \cdots,\ 2,\ 1,\ 0,\ -1,\ -2,\ -3,\ \cdots$$
is limited to the right.

Ex. 2. $$A = -5,\ -4,\ -3,\ -2,\ -1,\ 0,\ 1,\ 2,\ 3,\ \cdots$$
is limited to the left.

253. 1. It is sometimes convenient to divide an interval into equal subintervals or a square into equal subsquares, and, in general, an m-dimensional cube Γ into equal subcubes.

For $m = 1, 2, 3$, this needs no explanation. When m is > 3, the matter is still very simple. The cube Γ is graphically represented by m equal segments on the $x_1 \cdots x_m$ axes. We divide Γ into cubes whose sides are $1/n$th those of Γ by dividing each of these segments into n equal parts. One of these subcubes is then represented by the points which fall in a set of m segments as $\alpha_1\beta_1 \cdots \alpha_m\beta_m$.



2. Instead of a cube Γ in $\mathfrak{R}_m$, we may wish to divide the whole of $\mathfrak{R}_m$ into cubes. The meaning of this is now evident.

3. Let A be any point aggregate in $\mathfrak{R}_m$. Let us divide $\mathfrak{R}_m$ into cubes of side δ. This also, in general, divides A into partial aggregates. This division of A into partial aggregates we shall call a *cubical division of* A, *of norm* δ.

4. If instead of dividing $\mathfrak{R}_m$ into cubes, we had divided it into rectangular parallelopipeds whose edges are $\overline{\gtrless}\ \delta$, we shall say that we have effected a *rectangular division of* $\mathfrak{R}_m$, *of norm* δ.

5. The partial aggregates, into which A falls after a cubical or rectangular division, may also be called *cells*.

Limiting Points

254. 1. One of the most important notions connected with point aggregates is that of a *limiting point*. Let A be a point aggregate in $\mathfrak{R}_m$. Any point p of $\mathfrak{R}_m$ is a limiting point of A, if however small $\rho > 0$ is taken, $D_\rho(p)$ contains an infinity of points of A.

If every domain of p *contains at least one other point,* p *is a limiting point of* A.

For, let a be a point of A different from p. Let $0<\rho<\text{Dist}(p, a)$. Then, by hypothesis, $D_\rho(p)$ contains some points a_1 of A, besides p. Let $0 < \rho_1 < \text{Dist}(p, a_1)$. Then $D_{\rho_1}(p)$ contains some point a_2 of A, besides p. Continuing in this way, we see that the infinite aggregate of distinct points

$$a_1,\ a_2,\ a_3,\ \cdots$$

all lie in $D_\rho(p)$.

2. The following may also be taken as definitions of a limiting point:

If $V_\rho(p)$ *is infinite, however small* ρ *is taken,* p *is a limiting point of* A*; or,*

If $V_\rho^*(p) > 0$*, however small* ρ *is taken,* p *is a limiting point of* A.

3. If p is a limiting point of A and p itself lies in A, it is called a *proper limiting point.* If p is not in A, it is called an *improper limiting point.*

Any point of an aggregate A which is not a limiting point is an *isolated point.*

4. Let A be a rectilinear aggregate, and a one of its limiting points. If no point of A falls in $(a^*, a+\delta)$ or in $(a-\delta, a^*)$, $\delta > 0$ sufficiently small, a is called a *unilateral* limiting point. Otherwise a is a *bilateral* limiting point.

255. Ex. 1. $$A = 1, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4}, \cdots$$

Here the origin is a unilateral limiting point of A. As 0 does not lie in A, it is an improper limiting point.

Ex. 2. $$A = 0, 1, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4}, \cdots$$

The origin is a proper unilateral limiting point of A.

Ex. 3. $A =$ totality of rational numbers.

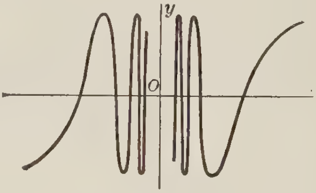
Every point p in $\mathfrak{R}$ is a bilateral limiting point.

If p is a rational point, it is a proper limiting point of A. If p is an irrational point, it is an improper limiting point.

Limiting Points connected with Certain Functions

256. We give now a few examples of point aggregates which come up in the study of certain functions.

Let $$y = \sin \frac{1}{x}.$$



The domain of definition of this function embraces all points on the x-axis except $x = 0$.

It oscillates between -1 and $+1$.

The points $$x = a_1, a_2, a_3, \cdots$$

for which y takes on a particular value, as $y = 0$, form a point aggregate whose limiting point is $x = 0$.

In any domain of this point, y oscillates from $+1$ to -1 an infinite number of times.

257. Let

$$y = \sin \frac{1}{\sin \frac{1}{x}}. \tag{1}$$

When $x = 0$, or when

$$\sin \frac{1}{x} = 0, \tag{2}$$

y is not defined, since for these points, 1) involves division by 0.

The points x for which 2) holds are

$$\pm \frac{1}{\pi}, \quad \pm \frac{1}{2\,\pi}, \quad \pm \frac{1}{3\,\pi}, \cdots \tag{3}$$

This is a point aggregate whose limiting point is $x = 0$.

As x approaches one of the points $\frac{1}{n\pi}$, y oscillates with increasing rapidity. At the same time these points, $\frac{1}{n\pi}$, become infinitely dense as x nears the origin. The domain of definition of y is the x-axis except the origin and the points 3).

258. Let

$$y = \sin \frac{1}{\sin \frac{1}{\sin \frac{1}{x}}}.$$

This expression does not define y, because of division by 0, when

$$x = 0, \tag{1}$$

or when x satisfies

$$\sin \frac{1}{x} = 0, \tag{2}$$

or

$$\sin \frac{1}{\sin \frac{1}{x}} = 0. \tag{3}$$

The points x defined by 1) and 2) are

$$A = 0, \quad \pm \frac{1}{\pi}, \quad \pm \frac{1}{2\,\pi}, \quad \cdots$$

considered in 257.

It is easy to see that the points x defined by 3) form an aggregate B such that each of the points of A is a limiting point of B. In fact, let x approach the point $\pm\frac{1}{n\pi}$. As it does so, $\sin\frac{1}{x}$ becomes smaller and smaller; hence $\frac{1}{\sin\frac{1}{x}}$ becomes larger and larger.

Thus in the domain of the point $\pm\frac{1}{n\pi}$,

$$\sin\frac{1}{\sin\frac{1}{x}}$$

oscillates infinitely often between -1, 1, and in particular 3) is satisfied infinitely often.

Thus, the domain of definition of y includes all points of the x-axis except the points A and B.

About each point of B, y oscillates infinitely often. These points of infinitely frequent oscillation, themselves cluster infinitely thick about each point of A; while the points A cluster infinitely dense about the origin. Let the reader try to picture to himself how the graph of y looks about the points

$$\pm\frac{1}{n\pi},\text{ and } 0.$$

259. 1. The functions of 257 and 258 are formed from that of 256 by a *process of iteration.*

In fact, let

$$y=\sin\frac{1}{x}=\theta(x);$$

then

$$\sin\left\{\frac{1}{\sin\frac{1}{x}}\right\}=\theta[\theta(x)].$$

Similarly,

$$\sin\left\{\frac{1}{\sin\frac{1}{\sin\frac{1}{x}}}\right\}=\theta\{\theta[\theta(x)]\},\text{ etc.}$$

It is customary to write

$$\theta^2(x) \text{ for } \theta[\theta(x)],$$

$$\theta^3(x) \text{ for } \theta[\theta^2(x)], \text{ etc.}$$

2. As another example of iteration, let

$$y = \log x = \theta(x).$$

Then

$$\theta^2(x) = \log(\log x),$$

$$\theta^3(x) = \log[\log(\log x)], \text{ etc.}$$

260. Let

$$y = \sin\frac{1}{x} = \theta(x).$$

We have noted the domain of definition of

$$\theta(x),\ \theta^2(x),\ \theta^3(x).$$

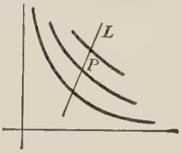
In general, let A_m be the domain of definition of $\theta^m(x)$. Each point of A_{m-1} is a limiting point of A_m, and $\theta^m(x)$ oscillates infinitely often about each point of A_m.

261. To get functions of two variables having more complicated domains of definition, we may apply the process of iteration to the function of 237.

Let

$$\theta(xy) = \tan\tfrac{1}{2}\pi xy.$$

We saw θ was not defined for points on the family of hyperbolas



$$H) \qquad xy = 2m + 1. \qquad m = 0,\ \pm 1,\ \pm 2,\ \cdots$$

Let us consider the domain of definition D_2 of

$$\theta^2(xy) = \tan(\tfrac{1}{2}\pi \tan\tfrac{1}{2}\pi xy).$$

Through any point P of one of these hyperbolas H) pass an arbitrary right line L.

At any point Q on the line, such that

$$\theta(xy) = 2n + 1, \qquad n = 0,\ \pm 1, \cdots$$

θ^2 is not defined.

But the points Q have P as limiting point.

Derivatives of Point Aggregates

262. 1. Let A be a point aggregate. The limiting points of A, if it has any, form an aggregate, which is called the *first derivative* of A, and is denoted by A'.

Ex. **1.**

$$A = \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \frac{6}{5}, \cdots = \left\{\frac{n+1}{n}\right\};$$

$$A' = 1.$$

Ex. **2.**

$$A = \frac{1}{2}, \frac{3}{2}, \frac{1}{3}, \frac{4}{3}, \frac{1}{4}, \frac{5}{4}, \frac{1}{5}, \frac{6}{5}, \cdots = \left\{\frac{1}{n}, \frac{n+1}{n}\right\};$$

$$A' = 0, 1.$$

2. When A is a finite aggregate,

$$A' = 0.$$

But A may be infinite and yet have no derivative.

Ex. **3.**

$$A = 1, 2, 3, 4, \cdots$$

is such an aggregate.

263. 1. The first derivative A' may have limiting points; their aggregate is called the *second derivative* of A. It is denoted by A''.

A'' may have limiting points; these give rise to the *third derivative* A''', etc.

Ex. **1.**

$$A = \left\{\frac{1}{m} + \frac{1}{n}\right\}; \qquad m, n = 1, 2, 3, \cdots$$

Let m be arbitrary but fixed; then $\frac{1}{m}$ is a limiting point of A. For A contains the points

$$\frac{1}{m} + 1, \quad \frac{1}{m} + \frac{1}{2}, \quad \frac{1}{m} + \frac{1}{3}, \quad \frac{1}{m} + \frac{1}{4}, \quad \cdots$$

whose limiting point is obviously $\frac{1}{m}$.

Thus,

$$A' = \left\{0, \frac{1}{m}\right\}; \qquad m = 1, 2, 3, \cdots$$

while

$$A'' = (0), \tag{1}$$

and

$$A''' = 0. \tag{2}$$

As explained in 243, 3, the equation 1) means that A'' consists only of the origin, while 2) indicates that A''' has no points at all.

Ex. 2.

$A =$ rational points in an interval I;

$A' = I.$ See 255, Ex. 3.

$A'' = I, \ A''' = I, \ \cdots$

Thus A has derivatives of every order, each being I.

2. If $A, A', \cdots A^{(m)} > 0$, while $A^{(m+1)} = 0$, A is of *order m*.

264. *Every limited infinite point aggregate has at least one limiting point.*

1. For simplicity let us consider first the case that the aggregate A lies in the interval $I = (a, b)$. We divide I into halves. One of these halves, call it I_1, contains an infinity of points of A. Divide I_1 in halves. One of these halves must contain an infinity of points of A. In this way we may continue bisecting each successive interval, without end. We get thus an infinite sequence of intervals

$$I, \ I_1, \ I_2, \ \cdots \qquad (1$$

each lying in the preceding, whose lengths converge to 0.

By 127, 2, the sequence 1) determines a point α. This point α lies in every interval of 1). Since each $D(\alpha)$ contains some I_n, it contains an infinity of points of A. Hence α is a limiting point of A.

2. The extension of this demonstration to $\mathfrak{R}_n$ is now readily made. Since A is limited, it lies in a certain parallelopiped P, viz.,

$$a_1 \le x_1 \le b_1 \ \cdots \ a_n \le x_n \le b_n,$$

by 252. We divide now P into two parts

$$a_1 \le x_1 \le \tfrac{1}{2}(b_1 - a_1), \ a_2 \le x_2 \le b_2 \ \cdots \ a_n \le x_n \le b_n, \qquad (2$$

and

$$\tfrac{1}{2}(b_1 - a_1) \le x_1 \le b_1, \ a_2 \le x_2 \le b_2 \ \cdots \ a_n \le x_n \le b_n.$$

In one of these there must lie an infinity of points of A. To fix the ideas suppose it is the parallelopiped 2) which we call Q_1.

Q_1 differs from P only in having one coördinate, viz. x_1, restricted to an interval half as big as the original.

We now divide Q_1 into two parts,

$$a_1 \le x_1 \le \tfrac{1}{2}(b_1 - a_1), \ a_2 \le x_2 \le \tfrac{1}{2}(b_2 - a_2), \ a_3 \le x_3 \le b_3, \ \cdots \qquad (3$$

and

$$a_1 \le x_1 \le \tfrac{1}{2}(b_1 - a_1), \ \tfrac{1}{2}(b_2 - a_2) \le x_2 \le b_2, \ a_3 \le x_3 \le b_3 \ .$$

In one of these, say it is the parallelopiped Q_2 defined by 3), an infinity of points of A must lie. Q_2 differs from P in having now two of its coördinates restricted to intervals only half as large as the original ones.

We may continue in this way restricting the remaining coördinates $x_3 \cdots x_n$, to intervals half as big as the original ones. We get parallelopipeds Q_3, $Q_4 \cdots Q_n$, in each of which lie an infinity of points of A.

Let us set $P_1 = Q_n$. This parallelopiped lies in P and has each edge just half as big as the corresponding edge of P.

We may now subdivide P_1 just as we did P. After n bisections we get a parallelopiped P_2, which lies in P_1, which contains an infinity of points of A, and whose edges are one half as big as those of P_1.

Continuing this process indefinitely, we get a sequence of parallelopipeds

$$P, \ P_1, \ P_2, \ \cdots$$

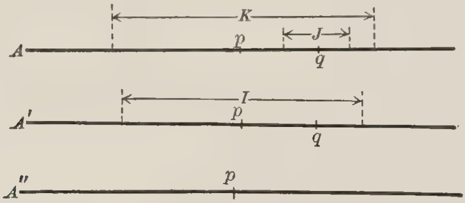
which determines a point $\alpha = (\alpha_1 \alpha_2 \cdots \alpha_n)$. This point is evidently a limiting point of A by the same reasoning as employed in 1.

265. *If A is a limited aggregate of the nth order, $A^{(n)}$ is finite.*

For if $A^{(n)}$ were infinite, being limited, it must have at least one limiting point, by 264. Then $A^{(n+1)} > 0$, which contradicts the hypothesis.

266. *Let A be any point aggregate. Then $A'' \leq A'$, i.e. all the limiting points of A' are proper.*

1. For simplicity, let us first consider a rectilinear aggregate.



Let p be any point of A''; we lay it off on the A, A' axes also as in the figure.

To show that p lies in A', we have to show it is a limiting point of A, *i.e.* in any little interval K about p there lie an infinity of points of A. Let I be any little interval about p on the A' axis. As p is a limiting point of A', I contains an infinity of points of A'. Let q be one of these. Let us lay q off on the A axis. Since q is a limiting point of A, any little interval as J contains an infinity of points of A. Now, however small K is taken, there exist intervals J lying within K which contain an infinity of points of A. Hence K contains an infinity of points of A, and p is a limiting point of A. Thus p lies in A'.

2. The extension of this demonstration to $\mathfrak{R}_m$ is obvious. To show that p lies in A', we have to show that $D_\epsilon(p)$ contains an infinity of points of A, however small ϵ is taken. To this end, let $\rho < \epsilon$. Let q be a point of A' in $D_\rho(p)$. Then $D_\sigma(q)$ contains an infinity of points of A, however small σ is. But if

$$\rho + \sigma < \epsilon,$$

$D_\sigma(q)$ lies in $D_\epsilon(p)$, by 249. Hence $D_\epsilon(p)$ contains an infinity of points of A.

3. We have just shown that A'' lies in A'. It is, however, not necessary that A' lies in A.

Thus, if $A = 1, \frac{1}{2}, \frac{1}{3}, \cdots$, $A' = (0)$, and this does not lie in A.

267. *Extreme values of a domain.* 1. Let the variable x range over a rectilinear domain D which is limited to the right.

We form a partition (A, B) as follows: in A we put all numbers of $\mathfrak{R}$ which are $\leqq$ any number in D; in B we put all numbers of $\mathfrak{R}$ which are $>$ any number of D.

Let this partition be generated by μ [130].

We call μ the *maximum of* x or *of* D, and write

$$\mu = \text{Max}\, x = \text{Max}\, D.$$

The fact that a domain E is *not limited to the right* may be denoted by

$$\text{Max}\, x = \text{Max}\, E = +\infty,$$

where x ranges over E.

2. Let D be limited to the left. We form a partition (A, B) by putting in A all numbers of $\mathfrak{R}$ which are $<$ any number of D,

and in B all numbers which are $\overset{=}{>}$ any number of D. If the number λ generates this partition, we call λ the *minimum of* x or *of* D, and write

$$\lambda = \text{Min}\, x = \text{Min}\, D.$$

The fact that a domain E is *not limited to the left* may be denoted by

$$\text{Min}\, x = \text{Min}\, D = -\infty,$$

where x ranges over E.

Ex. 1. $$D = (a,\ b),\ a < b.$$

$$\text{Min}\, x = a. \qquad \text{Max}\, x = b.$$

We note that x takes on both its minimum and maximum values in D.

Ex. 2. $$D = (0,\ 1,\ \tfrac{1}{2},\ \tfrac{2}{3},\ \tfrac{1}{3},\ \tfrac{3}{4},\ \tfrac{1}{4},\ \tfrac{4}{5},\ \cdots).$$

$$\text{Min}\, x = 0. \qquad \text{Max}\, x = 1.$$

Here x takes on both its maximum and minimum values.

Ex. 3. $$D = (a^*,\ b^*).$$

$$\text{Min}\, x = a. \qquad \text{Max}\, x = b.$$

Ex. 4. $$D = (\tfrac{1}{2},\ \tfrac{2}{3},\ \tfrac{1}{3},\ \tfrac{3}{4},\ \tfrac{1}{4},\ \tfrac{4}{5},\ \cdots).$$

$$\text{Min}\, D = 0. \qquad \text{Max}\, D = 1.$$

In Exs. 3, 4, x takes on neither its minimum nor its maximum values.

268. 1. The maximum and minimum values of x are called its *extreme values or extremes.*

Let e be an extreme of D. If the point e is an isolated point of D, e is called an *isolated extreme*, otherwise e is a *non-isolated extreme.*

Evidently *an isolated extreme of D lies in D.*

2. When, however, e is a non-isolated extreme, it may or may not lie in D. In this case we have the theorem:

If e be a finite non-isolated extreme of D, it is an extreme of D', the first derivative of D.

To fix the ideas, let

$$e = \text{Max}\, D.$$

Since e is not isolated, it is a limiting point of D, and hence lies in D'.

Since no x of D is $> e$, no x of D' is $> e$. Hence

$$e = \text{Max}\, D'.$$

3. We have obviously the following:

Let every x of D be $\overline{\gtrless}\, \mu$, *while for each* $\epsilon > 0$ *there exists in* D *an* $x > \mu - \epsilon$. *Then*

$$\mu = \text{Max}\, D.$$

A similar theorem holds for a *minimum.*

4. Let $\mathfrak{M}$ be such that

$$\text{Min}\, x \leq \mathfrak{M} \leq \text{Max}\, x;$$

we call $\mathfrak{M}$ *a mean value of* x, or *a mean of* x, and write

$$\mathfrak{M} = \text{Mean}\, x.$$

269. 1. *Let* e *be an extreme, finite or infinite, of the function* $f(x_1 \cdots x_m)$ *with respect to a limited domain* D. *Then there exists a point* a, *not necessarily in* D, *such that* e *is the extreme of* f *in any vicinity of* a, *however small.*

The demonstration is precisely analogous to that given in 264.

2. *If* D *contains its limiting points, the point* a *lies in* D.

Various Classes of Point Aggregates

270. Each point p of an aggregate A is either an isolated point or a limiting point of A. Let us call A_ι the aggregate of the former points and A_λ the aggregate of the latter points.

Then

$$A = A_\iota + A_\lambda.$$

If $A_\lambda = 0$, then $A = A_\iota$, and A is an *isolated* aggregate.

If $A_\iota = 0$, then $A = A_\lambda$, and A is *dense*.

A may contain *all* its limiting points; it is then *complete.*

If A is dense and complete, it is *perfect.* It then contains all its limiting points, and every point of A is a limiting point.

A point aggregate such that each of its points is an inner point is called a *region.* Cf. 247.

It is sometimes convenient to consider the aggregate formed of a region and its frontier points. Such an aggregate is called a *complete region.*

For example, the interior of a circle forms a region. If we add its circumference we get a complete region.

271. Ex. 1. $$A = 1, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4}, \cdots$$
is an *isolated* aggregate.

Ex. 2. $$A = 0, 1, \tfrac{1}{2}, \tfrac{1}{3}, \cdots$$
is a *complete* aggregate.

Ex. 3. $A =$ the rational numbers in a certain interval.
A is *dense*, but *not complete.*

Ex. 4. $A =$ the interval (a, b).
A is *perfect.*

Ex. 5. $A =$ the interval (a^*, b).
A is *dense*, but *not complete.*

Ex. 6. A region is *dense*, but not *complete.*
A completed region is perfect.

Ex. 7. In the interval $(0, 1)$ remove the points $0, 1, \tfrac{1}{2}, \tfrac{1}{3}, \cdots$
The remaining points form a *region.*

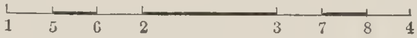
Ex. 8. In the plane $\mathfrak{R}_2$ let
$$A = a_1, \ a_2, \ \cdots$$
be an aggregate having a single limiting point a. Let us suppose that a does not lie in A. About each a_n let us describe a circle C_n of radius so small that no two circles have a point in common. The aggregate formed of the points of $\mathfrak{R}_2$ *within* each circle C_n is a region Γ_n.

The aggregate formed of all the regions Γ_n is also a *region.*

272. 1. An interesting example of a *rectilinear perfect* aggregate lying in an interval $\mathfrak{A}$ and yet not embracing all the points of $\mathfrak{A}$ is the following, due to Cantor.

Let
$$a = \cdot\, a_1 a_2 a_3 \cdots$$
be expressed in the triadic system [144], restricting, however, the numbers $a_1, a_2, \cdots$ to the values 0, 2. Then $A = \{a\}$ is such an aggregate, as we now show.

We can get a good idea of this aggregate as follows.



Let the interval (1, 4) be of unit length. We divide it into three equal segments, (1, 2), (2, 3), (3, 4). The points 1, 3 are points of A. No point of A falls within the middle segment (2, 3). We have therefore marked this segment heavy in the figure. We now divide the segments (1, 2) and (3, 4) into three equal segments,

$$(1, 5),\ (5, 6),\ (6, 2),\ \text{and}\ (3, 7),\ (7, 8),\ (8, 4).$$

The end points 6, 8 are points of A. No point of A falls within the segments (5, 6), (7, 8), which are therefore marked heavy.

In this way we can continue indefinitely subdividing the segments within which a point of A falls. Consider the end points of a heavy interval, say the interval (5, 6). Its right hand end point is obviously a number of A having a finite representation. Its left hand end point is a point of A whose representation is infinite.

To show now that A is perfect, let us begin by showing that every point a of A is a limiting point.

Let

$$a = \cdot\, a_1 a_2 \cdots a_s$$

be a point having a finite representation [144, 5].

Obviously, a is the limit of the sequence,

$$a' = \cdot\, a_1 \cdots a_s 2,$$

$$a'' = \cdot\, a_1 \cdots a_s 02,$$

$$a''' = \cdot\, a_1 \cdots a_s 002,$$

$$\cdots$$

Let

$$a = \cdot\, a_1 a_2 a_3 \cdots$$

be a point whose representation is infinite. It is obviously the limit of the sequence,

$$a' = \cdot\, a_1,$$

$$a'' = \cdot\, a_1 a_2,$$

$$a''' = \cdot\, a_1 a_2 a_3,$$

$$\cdots$$

Hence every point of A is a limiting point. On the other hand A contains all its limiting points.

For every limiting point α of A is either, 1°, an end point of the black intervals, or, 2°, not such a point.

In case 1°, α is obviously a point of A.

In case 2°, if $\alpha \neq 0$, we can find a monotone sequence of points in A.

$$a' = \cdot\, a_1,$$

$$a'' = \cdot\, a_1 a_2,$$

$$a''' = \cdot\, a_1 a_2 a_3,$$

$$\cdots$$

whose limit is α.

On the other hand, the limit of this sequence is

$$\cdot\, a_1 a_2 a_3 \cdots$$

which is a number of A. Hence, α is in A.

Should $\alpha = 0$, this method is inapplicable.

But obviously $\alpha = 0$ is in A.

2. It is easy to generalize the above example as follows. Let

$$a = \cdot\, a_1 a_2 a_3 \cdots$$

be expressed in an m-adic system restricting the numbers $a_1, a_2, \cdots$ to a part of the system,

$$0, 1, 2, \cdots m-1;$$

for example,

$$0, 2, 4, 6, \cdots$$

CHAPTER VI

LIMITS OF FUNCTIONS

FUNCTIONS OF ONE VARIABLE

Definitions and Elementary Theorems

273. 1. We extend now the notion of limit, by defining limits of functions. We begin by considering functions of a single variable x.

Let $f(x)$ be a one-valued function defined over a domain D. Let

$$A = a_1, \ a_2, \ a_3 \ \cdots \tag*{(1}$$

be *any* sequence of points of D, such that

$$\lim a_n = a; \qquad a \text{ finite or infinite}, \qquad a_n \neq a.$$

If the sequence

$$f(a_1), \ f(a_2), \ f(a_3) \ \cdots \tag*{(2}$$

has a limit η, finite or infinite, always the same, however the sequence A be chosen, we say *η is the limit of $f(x)$ for $x = a$* and write

$$\eta = \lim_{x=a} f(x),$$

or, more shortly,

$$\eta = \lim f(x).$$

We also say *$f(x)$ approaches* or *converges to η as a limit, when x approaches a as a limit.* This may be expressed by the symbol

$$f(x) \doteq \eta.$$

2. If for some sequence 1) the limit of 2) does not exist, we say the limit of $f(x)$ for $x = a$ does not exist.

3. Since the limit of 2) must be η however the sequences 1) are chosen (provided, of course, they have a as limit and $a_n \neq a$), we have the theorem :

Let $A = \{a_n\}$, $B = \{b_n\}$ *be two sequences lying in* D*; let* $a_n \doteq a$, $b_n \doteq a$.

If

$$\lim f(a_n) \neq \lim f(b_n),$$

then $\lim f(x)$, *for* $x = a$, *does not exist.*

274. 1. It is sometimes convenient to restrict the sequences $A = a_1, a_2, \cdots$ so that all the points a_n lie to the right of a. In this case we call η a *right hand limit* and write

$$\eta = \lim_{x=a+0} f(x) \text{ or } \eta = f(a+0) \text{ or } \eta = R \lim_{x=a} f(x) \text{ or } \eta = R \lim f(x).$$

If we restrict the sequences A to lie to the left of a, we call η a *left hand limit* and write

$$\eta = \lim_{x=a-0} f(x) \text{ or } \eta = f(a-0) \text{ or } \eta = L \lim_{x=a} f(x) \text{ or } \eta = L \lim f(x).$$

Obviously *if*

$$\lim f(x) = \eta, \quad \textit{finite or infinite.} \tag{1}$$

then

$$L \lim f(x) = R \lim f(x) = \eta. \tag{2}$$

Conversely, if 2) *holds,* 1) *does also.*

2. Right and left hand limits are called *unilateral* limits. If we do not care to specify on which side of a the limit is taken, we can denote it by

$$U \lim_{x=a} f(x).$$

275. 1. When considering infinite limits or limits for $x = \pm\infty$, it is often convenient to suppose the axes terminated to the right and left by two *ideal points* $+\infty$, or $-\infty$, respectively. We call these *the points at infinity.*

We call the interval $(G, +\infty)$ the domain of $+\infty$, and denote it by

$$D_G(+\infty). \tag{1}$$

We call G the norm of $D(+\infty)$.

Let A be a point aggregate lying on our axis. Those of its points which fall in 1) we call the vicinity of $+\infty$ for the aggregate A. We denote it by

$$V_G(+\infty). \qquad (2$$

Similar definitions hold for

$$D_G(-\infty) \text{ and } V_G(-\infty). \qquad (3$$

2. When G increases, the intervals $(G, +\infty)$ or $(G, -\infty)$ are, in a way, diminishing. It is convenient, for uniformity, to say that

$$D_G(\pm\infty),\ V_G(\pm\infty)$$

are arbitrarily small when G is taken arbitrarily large, positively or negatively, according to the sign of ∞.

276. Corresponding to the two ideal points $\pm\infty$, we shall introduce two *ideal numbers*, which we also denote by $\pm\infty$. These numbers are respectively greater, positively or negatively, than any number in $\mathfrak{R}$. We say they are *infinite.*

The system formed by joining $\pm\infty$ to the system $\mathfrak{R}$ we denote by $\overline{\mathfrak{R}}$.

We shall perform no arithmetical operations with these ideal numbers.

277. 1. Most of the theorems established in Chapters I, II for sequences may be extended easily to theorems on limits of functions.

For convenience of reference we collect the following. The reader should remember that a theorem relating to limits for a point $x=a$ may be changed at once into one relating to a left or a right hand limit at a.

2. *Let*

$$\lim_{x=a} f(x) = \alpha, \quad \lim_{x=a} g(x) = \beta. \qquad \textit{a finite or inf.}$$

Then

$$\lim\,(f \pm g) = \alpha \pm \beta,$$

$$\lim fg = \alpha\beta,$$

$$\frac{f}{g} = \frac{\alpha}{\beta}. \qquad \beta \neq 0.$$

See 49, 50, 51, 98.

3. *In $V^*(a)$, a finite or infinite, let*

$$f(x)\leq g(x)\leq h(x).$$

Let

$$\lim_{x=a} f(x)=\lim_{x=a} h(x)=\lambda.$$

Then

$$\lim g(x)=\lambda.$$

See 107.

4. *Let*

$$\lim_{x=a} f(x) \qquad \textit{a finite or inf.}$$

be finite. If

$$\lambda\leq f(x)\leq \mu, \qquad \textit{in } V^*(a)$$

then

$$\lambda\leq \lim_{x=a} f(x)\leq \mu.$$

See 106, 1.

5. *Let*

$$\lim_{x=a} f(x)=\alpha,\ \lim_{x=a} g(x)=\pm\infty. \quad \textit{a finite or inf.}$$

Then

$$\lim(f\pm g)=\pm\infty, \qquad \lim \frac{f}{g}=0.$$

If $\alpha\neq 0$,

$$\lim fg=\pm\infty.$$

See 137.

6. *Let*

$$\lim_{x=a} f(x)\neq 0,\ \lim_{x=a} g(x)=0. \qquad \textit{a finite or inf}$$

If $g(x)$ has one sign in $V^*(a)$,

$$\lim \frac{f}{g}=\pm\infty.$$

See 137.

7. *In $V^*(a)$, a finite or infinite, let*

$$f(x)\geqq g(x).$$

If

$$\lim_{x=a} g(x)=+\infty,$$

then

$$\lim_{x=a} f(x)=+\infty.$$

See 138.

8. *In $V^*(a)$ let $f(x)$ be limited and monotone. Then*

$$f(a+0), \qquad f(a-0)$$

exist and are finite.

See 109.

Second Definition of a Limit

278. 1. *If*

$$\lim_{x=a} f(x) = \eta, \qquad a \textit{ finite or inf.}$$

there exists for each $\epsilon > 0$ a vicinity $V^*(a)$ such that

$$|\eta - f(x)| < \epsilon \tag{1}$$

in $V^*(a)$.

For let D be the domain of definition of $f(x)$. Let $\Delta = \{\xi\}$ be the points of D, if any such exist, for which 1) is not satisfied. Let us suppose at first that a is finite. Let

$$\text{Min}\,|\xi - a| = \mu.$$

If $\mu > 0$, let $0 < \delta < \mu$, then 1) holds in $V_\delta^*(a)$.

If $\mu = 0$, let

$$\xi_1,\ \xi_2,\ \xi_3,\ \cdots$$

be a sequence in Δ, whose limit is a. Then

$$\lim_{n=\infty} f(\xi_n) \neq \eta,$$

and this contradicts the hypothesis.

Thus, when a is finite, there exists always a vicinity $V_\delta^*(a)$ for which 1) holds.

Suppose $a = +\infty$. Let

$$\text{Max}\,\xi = \mu.$$

If μ is finite, let $G > \mu$. Then 1) holds in $V_G(+\infty)$.

If $\mu = +\infty$, let

$$\xi_1,\ \xi_2,\ \xi_3,\ \cdots$$

be a sequence in Δ whose limit is $+\infty$. Then

$$\lim_{n=\infty} f(\xi_n) \neq \eta,$$

and this contradicts the hypothesis.

A similar reasoning applies when $a = -\infty$.

2. We wish expressly to note that in passing to the limit $x = a$, the variable x never takes on the value $x = a$.

279. The converse of the theorem 278 is obviously true, viz.:

If for each $\epsilon > 0$ *there exists a vicinity* $V_\delta^*(a)$, $\delta > 0$, *a finite or infinite, such that*

$$|\eta - f(x)| < \epsilon$$

in $V_\delta^*(a)$, *then*

$$\lim_{x=a} f(x) = \eta.$$

280. 1. From 278 and 279 we see that we can take the following as definitions of a limit:

The limit of $f(x)$ *for* $x = a$ *is* η *when for each* $\epsilon > 0$ *there exists a* $\delta > 0$, *such that*

$$|f(x) - \eta| < \epsilon \tag{1}$$

in $V_\delta^*(a)$.

This condition we shall express as follows:

$$\epsilon > 0, \quad \delta > 0, \quad |f(x) - \eta| < \epsilon, \quad V_\delta^*(a). \tag{2}$$

Such a line of symbols is to be read as above.

2. *The limit of* $f(x)$ *for* $x = +\infty$ *is* η, *when for each* $\epsilon > 0$ *there exists a* $G > 0$, *such that* 1) *holds in* $V_G(+\infty)$.

This condition we shall express thus:

$$\epsilon > 0, \quad G > 0, \quad |f(x) - \eta| < \epsilon, \quad V_G(+\infty).$$

3. *The limit of* $f(x)$ *for* $x = -\infty$ *is* η, *when for each* $\epsilon > 0$, *there exists a* $G < 0$, *such that* 1) *holds in* $V_G(-\infty)$.

This condition we shall express thus:

$$\epsilon > 0, \quad G < 0, \quad |f(x) - \eta| < \epsilon, \quad V_G(-\infty).$$

281. 1. *If*

$$\lim_{x=a} f(x) = +\infty, \quad a \text{ finite or infinite.}$$

there exists for each $G > 0$ *a vicinity* $V^*(a)$, *such that*

$$f(x) > G \tag{1}$$

in $V^*(a)$.

For, let $\Delta = \{\xi\}$ be the points of D, if any such exist, for which 1) does not hold.

1°. *Let a be finite.* Let

$$\text{Min}\,|\xi - a| = \mu.$$

If $\mu > 0$, let $0 < \delta < \mu$. Then 1) holds in $V_\delta^*(a)$.
If $\mu = 0$, let

$$\xi_1,\ \xi_2,\ \cdots$$

be a sequence in Δ whose limit is a. Then

$$\lim f(\xi_n) \neq +\infty\,;$$

and this contradicts the hypothesis.

2°. *Let $a = +\infty$.* Let

$$\text{Max}\,\xi = \mu.$$

If μ is finite, let $G > \mu$; then 1) holds in $V_G(+\infty)$.
If μ is infinite, let

$$\xi_1,\ \xi_2,\ \cdots$$

be a sequence in Δ whose limit is $+\infty$. Then

$$\lim f(\xi_n) \neq +\infty\,;$$

and this contradicts the hypothesis.

A similar reasoning holds when $a = -\infty$.

The reader will observe that this demonstration is analogous to that of 278.

2. The converse of 1) is obviously true, viz.:

If for each $G > 0$, there exists a vicinity $V^*(a)$, a finite or infinite, such that

$$f(x) > G$$

in $V^*(a)$, then

$$\lim_{x=a} f(x) = +\infty.$$

282. 1. From 281 we see that the following definitions of limits may be taken:

$$\lim_{x=a} f(x) = +\infty, \text{ if}$$

$$M > 0, \quad \delta > 0, \quad f(x) > M, \quad V_\delta^*(a),$$

which in full means: *if for each $M > 0$, large at pleasure, there exists a $\delta > 0$, such that $f(x) > M$ in $V_\delta^*(a)$.*

2. $$\lim_{x=a} f(x) = -\infty, \text{ if}$$

$$M<0, \quad \delta>0, \quad f(x)<M, \quad V_\delta^*(a);$$

i.e. if for each $M<0$ there exists a $\delta>0$, such that $f(x)<M$ in $V_\delta^*(a)$.

3. $$\lim_{x=+\infty} f(x) = +\infty, \text{ if}$$

$$M>0, \quad G>0, \quad f(x)>M, \quad V_G(+\infty).$$

4. $$\lim_{x=+\infty} f(x) = -\infty, \text{ if}$$

$$M<0, \quad G>0, \quad f(x)<M, \quad V_G(+\infty).$$

5. $$\lim_{x=-\infty} f(x) = +\infty, \text{ if}$$

$$M>0, \quad G<0, \quad f(x)>M, \quad V_G(-\infty).$$

6. $$\lim_{x=-\infty} f(x) = -\infty;$$

$$M<0, \quad G<0, \quad f(x)<M, \quad V_G(-\infty).$$

7. The limit, finite or infinite, of $f(x)$ for $x \doteq +\infty$ or $-\infty$ may be represented by

$$f(+\infty), \quad f(-\infty),$$

respectively.

283. By the aid of the ideal points, with their associate domains and vicinities, we may sum up all the preceding six cases in one general statement:

$$\eta = \lim_{x=a} f(x) \qquad a, \eta \textit{ finite or infinite,}$$

when, $D(\eta)$ being taken small at pleasure, there exists a vicinity $V^*(a)$, such that $f(x)$ lies in $D(\eta)$ when x runs over $V^*(a)$.

See 275, 2.

The reader should observe that the two demonstrations of 278 and 281 are perfectly parallel. It is easy, by employing the convention of 275, 2, to formulate the demonstration given in 278 so as to include the cases treated in 281, and so make the latter unnecessary.

284. *In order that* $\lim\limits_{x=a} f(x)$ *exists, a finite or infinite, it is necessary and sufficient that for each* $\epsilon > 0$ *there exists a vicinity* $V^*(a)$, *such that*

$$|f(x_1) - f(x_2)| < \epsilon, \tag{1}$$

for any pair of points x_1, x_2 *in* $V^*(a)$.

It is necessary. For, if

$$\eta = \lim_{x=a} f(x),$$

then for each $\epsilon > 0$ there exists a $V^*(a)$, such that

$$|\eta - f(x)| < \frac{\epsilon}{2}$$

for any x in $V^*(a)$. Let x_1, x_2 be two points in $V^*(a)$. Then

$$|\eta - f(x_1)| < \frac{\epsilon}{2}, \quad |\eta - f(x_2)| < \frac{\epsilon}{2}.$$

Adding these two inequalities, we get 1).

It is sufficient. For, let a_1, a_2, $\cdots$ be a sequence of points in $V^*(a)$, having a as limit. Then the sequence

$$f(a_1),\ f(a_2),\ \cdots$$

is regular by 1). It therefore has a limit η.

Then

$$\epsilon > 0, \quad m', \quad |\eta - f(a_n)| < \frac{\epsilon}{2}. \qquad n > m'. \tag{2}$$

Let $B = b_1$, b_2, $\cdots$ be *any* sequence of points in $V^*(a)$ whose limit is a. Then, by 1),

$$|f(a_n) - f(b_n)| < \frac{\epsilon}{2}. \qquad n > m''. \tag{3}$$

Adding 2), 3), we have

$$|\eta - f(b_n)| < \epsilon. \qquad n > m, \quad m > m', m''. \tag{4}$$

But since B was an arbitrary sequence, the relation 4) states that

$$\eta = \lim_{x=a} f(x).$$

Graphical Representation of Limits

285. The graphical representation of limits of sequences explained in 43, 44, and 124 may be readily extended to limits of functions.

Let the graph of $f(x)$ be referred to rectangular coördinates.

Let D be the domain of $f(x)$, and let

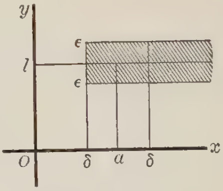
$$\lim_{x=a} f(x) = l.$$

Then the condition

$$\epsilon > 0,\ \delta > 0, \qquad |f(x) - l| \leq \epsilon, \qquad V_\delta^*(a),$$

has the following geometric interpretation:

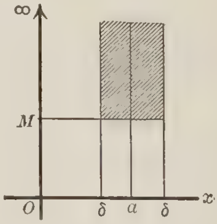
About the line $y = l$ construct a band (shaded in the figure) of width $2\,\epsilon$, ϵ being small at pleasure. Then there exists, corresponding to this ϵ, an interval of extent $2\,\delta$ (marked heavy in the figure), such that $f(x)$ falls in the ϵ-band for each $x \neq a$ of D falling in the δ-interval. In general, as ϵ is made smaller and smaller, δ becomes smaller and smaller. But for *each* ϵ-band, however small, there corresponds a δ-interval of length > 0.



286. Let

$$\lim_{x=a} f(x) = +\infty.$$

Draw the line $y = M$, where $M > 0$ is large at pleasure. Then there exists, corresponding to this M, a δ-interval, marked heavy in the figure, such that $f(x)$ falls in the M-band (shaded in the figure) for each $x \neq a$ of D, falling in the δ-interval.

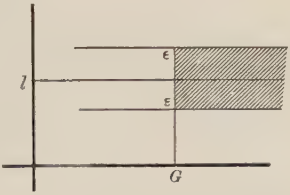


As M is taken greater and greater, the corresponding δ-interval becomes, in general, smaller and smaller. But for each M, however large, there corresponds a δ-interval of length > 0.

287. Let

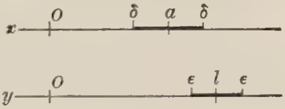
$$\lim_{x=+\infty} f(x) = l.$$

Draw the line $y = l$, and construct an ϵ-band, as in figure. For each ϵ there exists a $G > 0$, such that $f(x)$ falls in the ϵ-band for each x of D, falling in the interval $(G, +\infty)$.



These examples will suffice to illustrate the graphical interpretations of limits, when $f(x)$ is plotted in rectangular coördinates.

288. 1. When the graph of $y = f(x)$ is given by means of two axes, as explained in 191, the geometric interpretation of limits of $f(x)$ will be made clear by the following:



Let
$$\lim_{x=a} f(x) = l. \tag{1}$$

About $y = l$ we mark off the ϵ-interval; about $x = a$ we mark off the δ-interval.

Then 1) requires that $f(x)$ falls in the ϵ-interval for each value of $x \neq a$ in D, falling in the δ-interval.

2. Let
$$\lim_{x=a} f(x) = +\infty.$$

On the y-axis we mark off at pleasure the point $M > 0$. Then for each M there exists a δ-interval, such that $f(x)$ falls in the interval $(M, +\infty)$, for each $x \neq a$ of D falling in the δ-interval.

289. *Let*
$$\lim_{x=a} f(x) = l. \qquad \textit{l finite or infinite.}$$

$$x = a + bu, \qquad b \neq 0, \tag{1}$$

then
$$\lim_{u=0} f(x) = l. \tag{2}$$

For, while x ranges over the domain D on the x-axis, u ranges over a domain Δ on the u-axis.

The two axes x and u stand in 1 to 1 correspondence by virtue of 1). To the point $x = a$ on the x-axis corresponds the point $u = 0$ on the u-axis.

Let
$$f(x) = f(a + bu) = \phi(u). \qquad (3$$

Then if x and u are corresponding points, f has the same value at x as ϕ has at u. To fix the ideas let l be finite. From

$$\epsilon > 0, \quad \delta > 0, \quad |l - f(x)| < \epsilon, \quad \text{in } V_\delta{}^*(a),$$

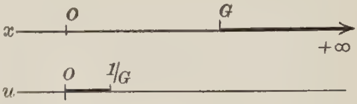
follows

$$|l - \phi(u)| < \epsilon, \quad \text{in } V_{\delta_1}{}^*(0), \qquad (4$$

where $\delta_1 = \frac{1}{b}\delta$.

But from 3), 4) follows 2).

290. 1. *Let*
$$\lim_{x=+\infty} f(x) = l. \qquad l \text{ finite or infinite.}$$
Let
$$x = \frac{1}{u}.$$
Then
$$R\lim_{u=0} f(x) = l,$$
and conversely.



This follows at once, as in 289, by observing that to points in the shaded interval on the x-axis correspond points in the shaded interval on the u-axis.

2. *Let*
$$\lim_{x=-\infty} f(x) = l. \qquad l \text{ finite or infinite.}$$
Let
$$x = \frac{-1}{u}.$$
Then
$$R\lim_{u=0} f(x) = l,$$
and conversely.

291. 1. As a result of 289 and 290, we may, by the aid of the transformations,
$$u = \alpha x + \beta, \quad \text{and} \quad u = \frac{1}{x},$$
transform
$$\lim_{x=a} \quad \text{into} \quad \lim_{u=b},$$
$$R\lim_{x=a} \quad \text{into} \quad R\lim_{u=b}, \quad L\lim_{u=b}, \quad \text{or} \quad \lim_{u=\pm\infty}.$$

Similarly,

$$\lim_{x=\pm\infty} \text{ into } R\lim_{u=a} \text{ or } L\lim_{u=a}.$$

2. In particular, any limit $x=a$ or $\pm\infty$ may be transformed into one with respect to $x=0$.

292. *Let* $u=\phi(x)$, *and*

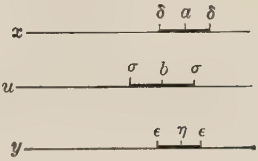
$$\lim_{x=a} u = b. \qquad a,\ b \textit{ finite or infinite.} \qquad (1$$

Let $y=f(u)$, *and*

$$\lim_{u=b} y = \eta. \qquad \eta \textit{ finite or infinite.} \qquad (2$$

Then if $\phi(x) \neq b$ *in* $V^*(a)$,

$$\lim_{x=a} y = \eta. \qquad (3$$



To fix the ideas, suppose a, b, η are finite. Then since 2) holds,

$$\epsilon>0, \quad \sigma>0, \quad |y-\eta|<\epsilon, \quad V_\sigma^*(b).$$

But, by 1),

$$\sigma>0, \quad \delta>0, \quad 0<|u-b|<\sigma, \quad V_\delta^*(a).$$

Hence while x ranges over $V_\delta^*(a)$, y lies in $D_\epsilon(\eta)$. Thus

$$\epsilon>0, \quad \delta>0, \quad |y-\eta|<\epsilon, \quad V_\delta^*(a).$$

But then 3) holds.

The case that any or all the symbols a, b, η are infinite is perfectly analogous.

293. *Let* $u=\phi(x)$, *and*

$$\lim_{x=a} u = b. \qquad a \textit{ finite or infinite.}$$

Let $y=f(u)$, *and*

$$\lim_{u=b} y = \eta.$$

If $f(b)=\eta$, *then*

$$\lim_{x=a} y = \eta.$$

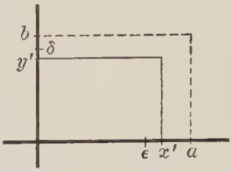
This follows as in 292.

294. 1. *Let* $y=f(x)$ *be a univariant function in a unilateral vicinity* V^* *of* a. *If*

$$U\lim_{x=a} y = b, \qquad [274,\ 2]$$

then

$$U\lim_{y=b} x = a.$$



To fix the ideas, suppose y is increasing in the left hand vicinity of a.

Let $\epsilon > 0$ be arbitrarily small, and

$$a - \epsilon < x' < a.$$

Let y' correspond to x'. Let $\delta > 0$ be such that

$$b - \delta > y'.$$

Then, while y remains in $LV_\delta^*(b)$, x remains in $LD_\epsilon(a)$.

2. *Let* $y=f(x)$ *be univariant in* $V^*(a)$, *where* a *is a bilateral limiting point* of V^*. *If*

$$\lim_{x=a} y = b,$$

then

$$\lim_{y=b} x = a.$$

The demonstration is analogous to that of **1**.

Examples of Limits of Functions

295. 1.
$$\lim_{x=0} \sin x = 0.$$

For, however small $\epsilon > 0$ is taken, there exists an arc $\delta > 0$ such that

$$|\sin x| < \epsilon. \qquad |x| < \delta.$$

2.
$$\lim_{x=0} \cos x = 1.$$

For, however small $\epsilon > 0$ is taken, there exists an arc $\delta > 0$ such that

$$1 - \cos x < \epsilon. \qquad |x| < \delta.$$

296. 1. $$\lim_{x=a} \sin x = \sin a. \qquad (1$$

For, let

$$x = a + u.$$

Then

$$\sin x = \sin (a + u) = \sin a \cos u + \cos a \sin u. \qquad (2$$

Since

$$\lim_{x=a} \sin x = \lim_{u=0} \sin (a + u),$$

and

$$\lim_{u=0} \sin u = 0, \ \lim_{u=0} \cos u = 1,$$

equation 2) gives, on passing to the limit, 1), by 289.

2. Similarly,

$$\lim_{x=a} \cos x = \cos a.$$

297. 1. $$L \lim_{x=\pi/2} \tan x = +\infty. \qquad (1$$

For, in $LV^*(\pi/2)$, $\tan x > 0$.

As

$$\tan x = \frac{\sin x}{\cos x},$$

and

$$\lim_{x=\pi/2} \sin x = 1, \ \lim_{x=\pi/2} \cos x = 0,$$

we have 1), by 277, 6.

2. Similarly,

$$R \lim_{x=\pi/2} \tan x = -\infty.$$

298. 1. $$\lim_{x=0} e^x = 1.$$

This follows at once from 172.

2. $$\lim_{x=a} e^x = e^a.$$

For, let

$$x = a + u.$$

Then, by 289,

$$\lim_{x=a} e^x = \lim_{u=0} e^{a+u} = e^a \lim_{u=0} e^u$$

$$= e^a, \text{ by } 1.$$

3.
$$\lim_{x=+\infty} e^x = +\infty.$$

This follows at once from 169.

4.
$$\lim_{x=-\infty} e^x = 0.$$

For, let
$$x = \frac{-1}{u}.$$

Then
$$\lim_{x=-\infty} e^x = R\lim_{u=0} \frac{1}{e^{\frac{1}{u}}}, \text{ by 290, 2;}$$
$$= 0, \text{ by 277, 5.}$$

5. Obviously, as in 1, 2,
$$\lim_{x=x_0} a^x = a^{x_0}.$$

6.
$$f(x) = \frac{e^{\frac{1}{x}} - 1}{e^{\frac{1}{x}} + 1}.$$
$$R\lim_{x=0} f(x) = +1, \qquad L\lim_{x=0} f(x) = -1.$$

299. 1. *Let*
$$\lim_{x=a} f(x) = \eta. \qquad \eta > 0.$$

Then
$$\lim_{x=a} (f(x))^\mu = \eta^\mu.$$

This follows directly from 171.

2. *In* $V^*(a)$, *let* $f(x) > 0$. *Let*
$$\lim_{x=a} f(x) = 0.$$

Then
$$\lim_{x=a} (f(x))^\mu = 0, \qquad \mu > 0.$$
$$= 1, \qquad \mu = 0.$$
$$= +\infty. \qquad \mu < 0.$$

300. 1.
$$\lim_{x=a} \log x = \log a. \qquad a > 0$$

This follows at once from 178.

2.
$$\lim_{a=+\infty} \log x = +\infty .$$

This follows at once from 179.

3.
$$R\lim_{x=0} \log x = -\infty .$$

For, set
$$x = \frac{1}{u} .$$

Then
$$R\lim_{x=0} \log x = \lim_{u=\infty} \log \frac{1}{u} = -\lim_{u=\infty} \log u = -\infty .$$

4. Let
$$\lim_{x=a} f(x) = \eta > 0 .$$

Then
$$\lim_{x=a} \log f(x) = \log \eta = \log \lim_{x=a} f(x) .$$

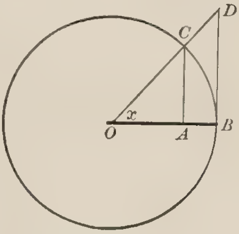
This follows at once from 178.

301. 1.
$$\lim_{x=0} \frac{\sin x}{x} = 1 .$$

From geometry, we have

$$\text{Area } OAC < \text{Area } OBC < \text{Area } OBD .$$

Hence, for $0 < x < \pi/2$,



$$\tfrac{1}{2} \sin x \cos x < \tfrac{1}{2} x < \tfrac{1}{2} \tan x ;$$

or
$$\cos x < \frac{x}{\sin x} < \frac{1}{\cos x} .$$

As
$$R\lim_{x=0} \cos x = R\lim_{x=0} \frac{1}{\cos x} = 1 ,$$

we have, by 277, 3,
$$R\lim_{x=0} \frac{x}{\sin x} = 1 . \qquad (1$$

Set in 1),
$$x = -u .$$

Then
$$R\lim_{x=0} \frac{x}{\sin x} = L\lim_{u=0} \frac{u}{\sin u} = 1 . \qquad (2$$

From 1), 2) we have

$$\lim_{x=0} \frac{x}{\sin x} = 1.$$

Whence, by 277, 2,

$$\lim_{x=0} \frac{\sin x}{x} = 1.$$

2. From 1 we have readily

$$\lim_{x=0} \frac{\sin ax}{bx} = \frac{a}{b}. \qquad b \neq 0.$$

For,

$$\frac{\sin ax}{bx} = \frac{a}{b} \cdot \frac{\sin ax}{ax}$$

$$= \frac{a}{b} \cdot \frac{\sin u}{u},$$

setting

$$u = ax.$$

302.
$$\lim_{x=0} \frac{\tan x}{x} = 1. \tag{1}$$

For,
$$\frac{\tan x}{x} = \frac{\sin x}{x} \cdot \frac{1}{\cos x}. \tag{2}$$

But
$$\lim_{x=0} \frac{\sin x}{x} = 1, \quad \lim_{x=0} \frac{1}{\cos x} = 1.$$

Thus, passing to the limit in 2), we get 1), by 277, 2.

303.
$$\lim_{h=0} \frac{\sin (x+h) - \sin x}{h} = \cos x. \tag{1}$$

For,
$$\frac{\sin (x+h) - \sin x}{h} = \frac{2 \cos (x + \frac{1}{2} h) \sin \frac{1}{2} h}{h}. \tag{2}$$

Also
$$\lim_{h=0} \cos (x + \tfrac{1}{2} h) = \cos x;$$

$$\lim_{h=0} \frac{2 \sin \frac{1}{2} h}{h} = \lim \frac{\sin \frac{1}{2} h}{\frac{1}{2} h} = 1.$$

Passing to the limit in 2), we get 1).

304. 1.
$$\lim_{x=0} \frac{1-\cos x}{x^2} = \frac{1}{2}. \tag{1}$$

For,
$$\frac{1-\cos x}{x^2} = \frac{2\sin^2 \frac{1}{2}x}{x^2} = \frac{1}{2}\left(\frac{\sin \frac{1}{2}x}{\frac{1}{2}x}\right)^2.$$

2.
$$\lim_{x=0} \frac{\tan x - \sin x}{x^3} = \frac{1}{2}.$$

For,
$$\frac{\tan x - \sin x}{x^3} = \frac{\tan x}{x} \cdot \frac{1-\cos x}{x^2}.$$

305. 1.
$$\lim_{x=+\infty} e^{-x} \cos x = 0. \tag{1}$$

Here
$$\lim e^{-x} = 0, \text{ by 298, 4,}$$

while
$$\lim \cos x$$

does not exist. We cannot, therefore, apply the theorem 277, 2, that the limit of the product is the product of the limits.

We therefore proceed thus:

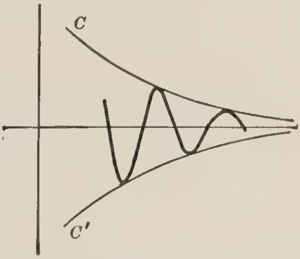
$$-e^{-x} \leq e^{-x} \cos x \leq e^{-x}.$$

Apply now 277, 3. This gives 1).

2. We may see the truth of 1) geometrically.

Let
$$y_1 = e^{-x},\ y_2 = \cos x,$$

and
$$y = e^{-x} \cos x = y_1 y_2.$$



Let us draw the graphs C, C' of $\pm y_1$. To get y, we multiply y_1 by the factor y_2, which takes on all values between -1 and $+1$. Thus y oscillates between the curves C, C'.

As C, C' approach nearer and nearer the x-axis, the amplitude of the oscillations converges to 0.

The Limit e and Related Limits

306. 1. We saw in 110 that

$$\lim_{n=\infty}\left(1+\frac{1}{n}\right)^n = e; \qquad n = 1, 2, 3, \cdots \tag*{(1}$$

Let us consider now the more general limit

$$\lim_{x=+\infty}\left(1+\frac{1}{x}\right)^x.$$

Each x will lie between two integers n, $n+1$; or

$$n \leq x < n+1.$$

Then

$$1+\frac{1}{n} \geqq 1+\frac{1}{x} > 1+\frac{1}{n+1},$$

and

$$\left(1+\frac{1}{n}\right)^{n+1} > \left(1+\frac{1}{x}\right)^x > \left(1+\frac{1}{n+1}\right)^n. \tag*{(2}$$

But

$$\left(1+\frac{1}{n}\right)^{n+1} = \left(1+\frac{1}{n}\right)\left(1+\frac{1}{n}\right)^n, \tag*{(3}$$

and

$$\left(1+\frac{1}{n+1}\right)^n = \left(1+\frac{1}{n+1}\right)^{n+1} \cdot \frac{1}{1+\dfrac{1}{n+1}}. \tag*{(4}$$

Thus 3), 4) give in 2),

$$\left(1+\frac{1}{n}\right)\left(1+\frac{1}{n}\right)^n > \left(1+\frac{1}{x}\right)^x > \left(1+\frac{1}{n+1}\right)^{n+1} \cdot \frac{1}{1+\dfrac{1}{n+1}}. \tag*{(5}$$

Now, by 1),

$$\lim\left(1+\frac{1}{n}\right)^n = \lim\left(1+\frac{1}{n+1}\right)^{n+1}.$$

Also,

$$\lim\left(1+\frac{1}{n}\right) = \lim\frac{1}{1+\dfrac{1}{n+1}} = 1.$$

Hence 5) gives

$$\lim\left(1+\frac{1}{x}\right)^x = e.$$

307.

$$\lim_{x=-\infty}\left(1+\frac{1}{x}\right)^x = e. \qquad (1$$

For, let

$$x = -u.$$

Then

$$\left(1+\frac{1}{x}\right)^x = \left(1-\frac{1}{u}\right)^{-u} = \left(1+\frac{1}{u-1}\right)^u$$

$$= \left(1+\frac{1}{v}\right)\left(1+\frac{1}{v}\right)^v, \qquad (2$$

where

$$u - 1 = v.$$

But, when $x \doteq -\infty$, $u \doteq +\infty$, and hence $v \doteq +\infty$.

As

$$\lim_{v=+\infty}\left(1+\frac{1}{v}\right) = 1, \quad \lim_{v=+\infty}\left(1+\frac{1}{v}\right)^v = e,$$

we get 1), on passing to the limit in 2).

308. From

$$\lim_{x=+\infty}\left(1+\frac{1}{x}\right)^x = e,$$

we get, setting

$$x = \frac{1}{u},$$

$$R\lim_{u=0}(1+u)^{\frac{1}{u}} = e. \qquad (1$$

From

$$\lim_{x=-\infty}\left(1+\frac{1}{x}\right)^x = e,$$

we get, setting

$$x = \frac{1}{u},$$

$$L\lim_{u=0}(1+u)^{\frac{1}{u}} = e. \qquad (2$$

From 1), 2) we have

$$\lim_{x=0}(1+x)^{\frac{1}{x}} = e. \qquad (3$$

309.

$$\lim_{u=0}(1+xu)^{\frac{1}{u}}=e^x. \qquad (1$$

For,

$$y=(1+ux)^{\frac{1}{u}}=\{(1+ux)^{\frac{1}{ux}}\}^x$$

$$=\{(1+v)^{\frac{1}{v}}\}^x,$$

where

$$v=ux.$$

But

$$\lim_{v=0}(1+v)^{\frac{1}{v}}=e.$$

Hence, by 299,

$$\lim y=e^x.$$

310.

$$\lim_{x=0}\frac{\log(1+x)}{x}=1.$$

For,

$$\lim_{x=0}\frac{\log(1+x)}{x}=\lim_{x=0}\log(1+x)^{\frac{1}{x}}$$

$$=\log\lim_{x=0}(1+x)^{\frac{1}{x}}, \text{ by 300, 4)}$$

$$=\log e, \text{ by 308, 3)}$$

$$=1.$$

311.

$$\lim_{x=0}\frac{a^x-1}{x}=\log a. \qquad a>0. \qquad (1$$

Set

$$u=a^x-1.$$

Then

$$\lim_{x=0}u=0,$$

by 298, 5. Then, by 292,

$$\lim_{u=0}\frac{\log(1+u)}{u}=\lim_{x=0}\frac{\log a^x}{a^x-1} \qquad (2$$

$$=1, \quad \text{by 310.}$$

But

$$\log a^x=x\log a.$$

This in 2) gives 1).

312.
$$\lim_{x=0} \frac{(1+x)^\mu - 1}{x} = \mu. \tag*{(1}$$

From 310 we have

$$\lim_{u=0} \frac{\log(1+u)}{u} = 1. \tag*{(2}$$

Let

$$u = (1+x)^\mu - 1. \qquad \mu \neq 0.$$

Then, by 299,

$$\lim_{x=0} u = 0.$$

Hence, by 292, we get from 2)

$$\lim_{x=0} \frac{\log(1+x)^\mu}{(1+x)^\mu - 1} = 1,$$

or

$$\lim \frac{(1+x)^\mu - 1}{\log(1+x)} = \mu, \tag*{(3}$$

since

$$\log(1+x)^\mu = \mu \log(1+x).$$

But, by 310,

$$\lim \frac{x}{\log(1+x)} = 1. \tag*{(4}$$

Now

$$\frac{(1+x)^\mu - 1}{\log(1+x)} = \frac{(1+x)^\mu - 1}{x} \cdot \frac{x}{\log(1+x)}.$$

Passing to the limit and using 4), we get 1) for the case that $\mu \neq 0$. The case that $\mu = 0$ is self-evident.

FUNCTIONS OF SEVERAL VARIABLES

Definitions and Elementary Theorems

313. For the sake of clearness, we have treated first the limits of functions of a single variable. We consider now the limits of functions in m variables. The extension of the definitions and results of the preceding sections is, for the most part, so obvious that we shall not need to enter into much detail. Should the reader have trouble with the case of general m, let him first suppose $m = 2$ or 3, when he can use his geometric intuition as a guide.

314. In the case of a single variable, we have seen how useful the ideal points ± ∞ proved. In the treatment of limits of functions of several variables, we shall find it extremely advantageous to adjoin an infinity of ideal points to $\mathfrak{R}_m$ as follows:

Let $A = a_1, a_2, a_3, \cdots$ be an infinite sequence of points in $\mathfrak{R}_m$. Let

$$a_s = (a_s', a''_s, \cdots a_s^{(m)}).$$

Let

$$\lim_{s=\infty} a_s' = \alpha_1, \cdots \lim_{s=\infty} a_s^{(m)} = \alpha_m;$$

$$\alpha_1, \cdots \alpha_m, \textit{ finite or infinite.}$$

We say the limit of the sequence A is

$$\alpha = (\alpha_1 \cdots \alpha_m), \qquad (1$$

and write

$$\alpha = \lim_{n=\infty} a_n.$$

If any of the coördinates of α are infinite, we say α is an infinite point. This fact may be briefly denoted by

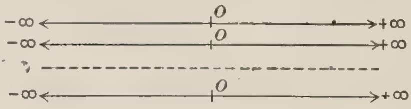
$$\alpha = \infty,$$

the symbol ∞ being without sign.

There is no point in $\mathfrak{R}_m$ corresponding to an infinite α. We therefore introduce an infinite system of ideal points, one for each complex,

$$\alpha_1, \alpha_2, \cdots \alpha_m, \qquad (2$$

in which one at least of the symbols, α_κ, is $\pm\infty$. Such ideal points we represent also by 1), and call the m symbols, 2), their coördinates. If we employ the graphical representation of 231, we suppose, according to 275, that each axis is terminated by the ideal points $+\infty$ and $-\infty$.



Thus, any complex of m points, one on each axis, such that at least one of these m points is an ideal point, is the representation of an ideal point in $\mathfrak{R}_m$.

The system of points, formed of $\mathfrak{R}_m$ and the ideal points, we denote by $\overline{\mathfrak{R}}_m$.

These ideal points are also called *points at infinity*.

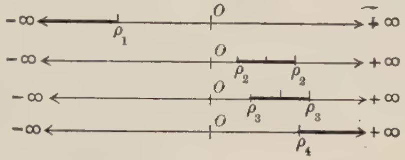
315. 1. The *domain of an ideal point* $a=(a_1\cdots a_m)$ is the aggregate of points $x=(x_1\cdots x_m)$, whose coördinates lie respectively in the domains

$$D_{\rho_1}(a_1), \cdots D_{\rho_m}(a_m).$$

It may be represented by $D_{\rho_1\cdots\rho_m}(a)$.

EXAMPLE. Let $m=4$, and $a=(-\infty, 1, 2, +\infty)$. The domains in which the coördinates x_1, x_2, x_3, x_4 range, are marked heavy in the figure.

Here ρ_1 is an arbitrarily large negative number; ρ_2 and ρ_3 are arbitrarily small positive numbers; ρ_4 is an arbitrarily large positive number.



2. The points of an aggregate A, which lie in $D_{\rho_1\cdots\rho_m}(a)$, a being a point at infinity, form the vicinity of a, for that aggregate. We represent it by

$$V_{\rho_1\cdots\rho_m}(a).$$

316. 1. Let $y=f(x_1\cdots x_n)$ be defined over a domain D. Let $A=a_1, a_2, \cdots$ be a sequence of points in D, and let

$$\lim a_n=\alpha. \qquad \alpha \textit{ finite or infinite.}$$

If

$$\lim_{n=\infty} f(a_n)=\eta, \qquad \eta \textit{ finite or infinite.}$$

is always the same however A be chosen, α remaining fixed and $a_n\neq\alpha$, we say *η is the limit of y for $x=\alpha$*; and write

$$\eta=\lim_{x_1=a_1\cdots x_m=a_m} f(x_1\cdots x_m),$$

or, more briefly,

$$\eta=\lim_{x=a} f(x_1\cdots x_m), \text{ or } \eta=\lim_{x=a} f(x);$$

or,

$$f(x_1\cdots x_m)\doteq\eta, \text{ or } f(x)\doteq\eta.$$

2. Just as in the case of a single variable, we can show that this definition is equivalent to the following:

$$\eta=\lim_{x=a} f(x_1\cdots x_m), \qquad \eta \textit{ finite or inf.}$$

when, taking $D(\eta)$ arbitrarily small, there exists a vicinity $V^*(\alpha)$, such that y remains in $D(\eta)$ when x is in $V^*(\alpha)$. See 278–283.

317. 1. The theorems of 277 and 284 hold for functions of several variables as well as for a single variable.

2. The generalized theorem of 292 may be stated thus:

Let

$$u_1 = \phi_1(x_1 \cdots x_n), \cdots u_m = \phi_m(x_1 \cdots x_n);$$

and

$$\lim_{x=a} u_1 = b_1, \cdots \lim_{x=a} u_m = b_m.$$

Let

$$y = f(u_1 \cdots u_m),$$

and

$$\lim_{u=b} y = \eta.$$

Let $u \neq b$ *in* $V^*(a)$. *Then*

$$\lim_{x=a} y = \eta.$$

Here a, b, η *may be finite or infinite.*

The demonstration is perfectly analogous to that in 292.

A Method for Determining the Non-Existence of a Limit

318. To determine whether

$$\eta = \lim_{x=a} f(x_1 \cdots x_m), \qquad a, \eta \textit{ finite or inf.}$$

even exists, is often a difficult matter. The following simple consideration analogous to 273, 2, 3 will sometimes show very easily that η does not exist. Let W be some partial vicinity of a excluding a. We may denote the limit, when it exists, of $f(x_1 \cdots x_m)$ for $x = a$ when x is restricted to W by

$$\zeta = \lim_{W} f(x_1 \cdots x_m).$$

Then ζ must exist, finite or infinite; and however W is taken, we must have

$$\eta = \zeta.$$

Thus, in case ζ does not exist, or is different for different W's, we know that η does not exist.

319. Ex. 1.

$$f(x, y)=\frac{xy}{x^2+y^2}.$$

We ask, does

$$\lim_{x, y=0} f$$

exist? As partial vicinity of the origin, take points on a line

L;
$$y = ax. \qquad x \neq 0.$$

Then

$$\lim_{L} f(x, y)=\lim_{x=0}\frac{ax^2}{x^2(1+a^2)}=\frac{a}{1+a^2},$$

which varies with a; *i.e.* with L.

Hence the limit in question does not exist.

320. Ex. 2.

$$f(x, y)=\frac{xy^2}{x^2+y^4}.$$

Does

$$\lim_{x, y=0} f \tag{1}$$

exist?

If we take as partial vicinity of the origin points on the line

L;
$$y = ax,$$

we get

$$\lim_{L} f(x, y)=\lim_{x=0} x\cdot\frac{a^2}{1+a^4x^2}=0. \tag{2}$$

Thus, however L is chosen, the limit 2) is always the same. We cannot, however, infer that the limit 1) exists, since our method only shows the non-existence of the limit.

Instead of the family of right lines L, let us take a family of parabolas

P;
$$y^2 = ax.$$

Then

$$\lim_{P} f(x, y)=\lim_{x=0}\frac{ax^2}{x^2(1+a^2)}=\frac{a}{1+a^2},$$

which varies with the particular parabola chosen.

Hence the limit 1) does not exist.

321. Ex. 3.

$$f(x, y)=\log\frac{a-x}{a-y}. \qquad x, y<a.$$

Does

$$\lim_{x, y=a} f$$

exist?

Let x, y lie on the line

L;
$$a-x=\lambda(a-y). \qquad \lambda>0.$$

Then

$$f(x, y)=\log\lambda.$$

Hence

$$\lim f(x, y)=\log\lambda,$$

which varies with λ.

Iterated Limits

322. 1. Let $f(x_1 \cdots x_m)$ be defined over some domain D; and let

$$a = (a_1 \cdots a_m).$$

Then

$$\lim_{x_{\iota_1}=a_{\iota_1}} f(x_1 \cdots x_m) = f_{\iota_1}$$

will be in general a function of all the variables except x_{ι_1}. Also

$$\lim_{x_{\iota_2}=a_{\iota_2}} f_{\iota_1} = f_{\iota_1\iota_2} = \lim_{x_{\iota_2}=a_{\iota_2}} \cdot \lim_{x_{\iota_1}=a_{\iota_1}} f(x_1 \cdots x_m)$$

will be in general a function of all the variables except x_{ι_1}, x_{ι_2}. Continuing, we arrive at

$$\lim_{x_{\iota_s}=a_{\iota_s}} \cdots \lim_{x_{\iota_2}=a_{\iota_2}} \cdot \lim_{x_{\iota_1}=a_{\iota_1}} f(x_1 \cdots x_m), \qquad s \leqq m, \qquad (1$$

which is in general a function of all the m variables except $x_{\iota_1}, x_{\iota_2}, \cdots x_{\iota_s}$.

Limits of the type 1) are called *iterated limits.*

In 1), we pass to the limit first with respect to x_{ι_1}, then with respect to x_{ι_2}, then with respect to x_{ι_3}, etc.

A change in the order of passing may produce a change in the final result.

2. Iterated limits occur constantly in the calculus; for example, in partial differentiation, differentiation under the integral sign, double integrals, improper integrals, and double series. The treatment of these subjects by the older writers on the calculus is faulty, as we shall see, because they change the order of passing to the limit, without a careful consideration of the correctness of such a step.

323. **Ex. 1.**

$$\lim_{y=0} \lim_{x=0} \frac{x-y}{x+y} = \lim_{y=0} \left(\frac{-y}{y}\right) = -1.$$

$$\lim_{x=0} \lim_{y=0} \frac{x-y}{x+y} = \lim_{x=0} \left(\frac{x}{x}\right) = +1.$$

The two limits are thus different.

Ex. 2.

$$\lim_{y=0} \lim_{x=\infty} \frac{1-xy}{1+xy} = -1.$$

$$\lim_{x=\infty} \lim_{y=0} \frac{1-xy}{1+xy} = +1.$$

The two limits are thus different.

324. The following is a case where a change in the order of passing to the limit does not change the result.

Let

$$\lim_{x=a,\, y=b} f(x, y) = \eta, \qquad \eta \textit{ finite or inf.} \tag{1}$$

$$\lim_{y=b} f(x, y) = g(x), \qquad \textit{for } 0 < |x - a| \leq \sigma. \tag{2}$$

$$\lim_{x=a} f(x, y) = h(y), \qquad \textit{for } 0 < |y - b| \leq \sigma. \tag{3}$$

Then

$$\lim_{x=a} g(x) = \lim_{y=b} h(y) = \eta. \tag{4}$$

Let η be finite. From 1) we have

$$\eta - \epsilon < f(x, y) < \eta + \epsilon, \qquad \text{in } V_\delta^*(a, b).$$

In this relation, pass to the limit $x = a$; then

$$\eta - \epsilon \leq g(x) \leq \eta + \epsilon, \quad \textit{for } 0 < |x - a| < \delta.$$

Hence

$$\lim g(x) = \eta. \tag{5}$$

Similarly,

$$\lim h(y) = \eta. \tag{6}$$

From 5), 6) we have 4).

Let $\eta = +\infty$. Then from 1)

$$f(x, y) > G, \qquad \text{in } V_\delta^*(a, b).$$

Passing to the limit, for $x = a$, we have

$$g(x) \geqq G, \qquad \textit{for } 0 < |x - a| < \delta.$$

Hence

$$\lim g(x) = +\infty = \eta.$$

Similarly,

$$\lim h(x) = +\infty = \eta.$$

These two equations give 4).

Uniform Convergence

325. 1. A notion of utmost importance in modern mathematics is that of uniform convergence. Let $f(x_1 \cdots x_m;\ t_1 \cdots t_n)$ be a function of two sets of variables, $x_1 \cdots x_m$ and $t_1 \cdots t_n$.

Let f be defined when $x = (x_1 \cdots x_m)$ runs over a domain D, and $t = (t_1 \cdots t_n)$ runs over Δ.

For each x in D, let

$$\lim_{t=\tau} f(x_1 \cdots x_m;\ t_1 \cdots t_n) = g(x_1 \cdots x_m).$$

Then for each $\epsilon > 0$ and each x in D there exists a $\delta' > 0$, such that

$$|f - g| < \epsilon \tag{1}$$

for any t in $V_{\delta'}^*(\tau)$.

Evidently if 1) holds for δ', it holds for any δ'', such that $0 < \delta'' < \delta'$. Of all the values δ' for which 1) holds at x, let δ be the maximum. Then for a given ϵ, δ is a well-defined function of x. In D, let

$$\text{Min } \delta = \delta_0.$$

Then $\delta_0 \gtreqless 0$. If, however small ϵ is taken, the corresponding δ_0 is > 0, we say *f converges uniformly to g in D*; or is *uniformly convergent.*

Hence, if f is uniformly convergent in D, there exists for each $\epsilon > 0$ a $\delta > 0$, such that

$$|f(x_1 \cdots x_m;\ t_1 \cdots t_n) - g(x_1 \cdots x_n)| < \epsilon$$

for any t in $V_\delta^*(\tau)$. Moreover, one and the same norm δ suffices for all the points of D, ϵ being the same.

The central idea of this case of uniform convergence may be clearly, if somewhat roughly, brought out by saying that if the convergence is uniform the norms δ for which 1) hold, ϵ being small at pleasure, but then fixed, do not sink below some definite positive number, when x ranges over D.

2. These considerations may be extended to the case that *τ is infinite;* we therefore define as follows:

The function $f(x_1 \cdots x_m;\ t_1 \cdots t_n)$ converges uniformly to $g(x_1 \cdots x_m)$ in D as $t \doteq \tau$, τ infinite; when for each $\epsilon > 0$, there exists a set of norms $\rho_1 \cdots \rho_n$, such that for any x in D,

$$|f(x_1 \cdots x_m;\ t_1 \cdots t_n) - g(x_1 \cdots x_m)| < \epsilon$$

in $V_{\rho_1 \cdots \rho_n}(\tau)$.

In this case of uniform convergence we may say: the norms ρ corresponding to infinite coördinates of τ cannot become infinitely great, and the norms corresponding to finite coördinates of τ cannot become indefinitely small as x ranges over D, for any given value of ϵ.

3. When $f(x_1 \cdots x_m;\ t_1 \cdots t_n)$ converges uniformly in D to $g(x_1 \cdots x_m)$, we denote this fact by

$$\lim f(x_1 \cdots x_m;\ t_1 \cdots t_n) = g(x_1 \cdots x_m),\quad \textit{uniformly.}$$

4. If $f \doteq 0$ uniformly in D, we may say it is *uniformly evanescent in D.*

326. Ex. 1.

$$f(x, t) = \frac{1}{x+t}.$$

$$D = (0^*, 1). \qquad \Delta = (-h, h). \qquad h > 0.$$

Evidently for any x in D

$$\lim_{t=0} f(x, t) = \frac{1}{x} = g(x).$$

But $f(x, t)$ does not converge uniformly to $g(x)$ in D. For if it did, for each $\epsilon > 0$ there must exist a $\delta > 0$, such that

$$R = |f(x, t) - g(x)| = \frac{|t|}{x\,|x+t|} < \epsilon \tag{1}$$

for any t in $V_\delta^*(0)$ and any x in D.

Now obviously, t being fixed, R can be made as large as we choose by taking x near enough 0. Hence R does not satisfy 1) as x ranges over D.

In fact, as is seen at once, in order to have $R < \epsilon$, it is necessary to take δ smaller and smaller as x approaches 0. In this case then

$$\text{Min}\ \delta = 0, \qquad \text{in } D.$$

327. Ex. 2.

$$f(x, t) = \frac{1}{x+t}.$$

$$D = (a, b),\ 0 < a < b. \qquad \Delta = (-h, h),\ h > 0.$$

This example is the same as Ex. 1, except D is different.

As before

$$\lim_{t=0} f(x, t) = \frac{1}{x} = g(x).$$

But now $f(x, t)$ converges uniformly to $g(x)$ in D.

In fact, in $V_\delta^*(0)$

$$R \leq \frac{\delta}{a(a-\delta)}, \qquad \delta < a,$$

wherever x is in D. But we can take δ, such that

$$\frac{\delta}{a(a-\delta)} < \epsilon.$$

Then

$$R < \epsilon$$

for any t in $V_\delta^*(0)$ and any x in D; *i.e.* f converges uniformly in D.

328. Ex. 3.

$$f(x, t) = 1 + x^2 - \frac{1}{(1 + x^2)^t}.$$

$$D = (-a, a). \qquad \Delta = (\alpha, +\infty).$$

Here

$$\lim_{t=\infty} f(x, t) = 1 + x^2, \qquad x \neq 0.$$

$$= 0. \qquad x = 0.$$

Hence if we set

$$g(x) = 0, \qquad \text{for } x = 0$$

$$= 1 + x^2, \qquad \text{for } x \neq 0,$$

we have

$$\lim_{t=\infty} f(x, t) = g(x).$$

However, f does not converge uniformly to g in D. For, when $x \neq 0$,

$$R = |f(x, t) - g(x)| = \frac{1}{(1 + x^2)^t}.$$

This shows that as x approaches 0, it is necessary to take t larger and larger in order that $R < \epsilon$.

There is thus no norm ρ, such that

$$R < \epsilon$$

for each $t > \rho$, and any x in D.

In this case, then,

$$\text{Max } \rho = \infty.$$

Remarks on Dirichlet's Definition of a Function

329. The definition of a function given in 189 and 230 does not depend at all upon an analytic expression for the function.

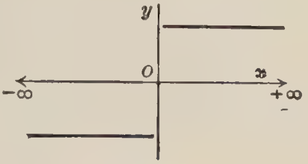
At first, the reader who has been used only to functions defined by analytic expressions, may be inclined to regard functions not thus defined as only *pseudo-functions*, or at least of little importance.

This attitude of mind must be overcome. In the first place, in certain parts of mathematical physics, *e.g.* the potential theory, it is of great importance to be able to assign values to a function at pleasure, totally disregarding the question of an analytic expression for it.

Secondly, as the reader advances, he will find that many functions which he might well believe have no analytic expression, do indeed have very simple ones.

We give now a few examples of such functions.

330. 1. For $x > 0$ let $y = 1$.
For $x = 0$ let $y = 0$.
For $x < 0$ let $y = -1$.



The graph of this function is given in the figure.

An analytic expression of y is

$$y = \frac{2}{\pi} \lim_{n=\infty} \operatorname{arctg}(nx).$$

This function is much used in the Theory of Numbers. We shall call it *signum* x and denote it by

$$y = \operatorname{sgn} x.$$

When $u, v \neq 0$ an equation

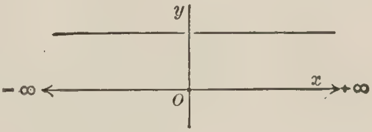
$$\operatorname{sgn} u = \operatorname{sgn} v$$

simply means that the sign of u is the same as that of v; while

$$\operatorname{sgn} u = +1$$

is only another way of saying that u is positive etc.

331. For $x \neq 0$ let $y = 1$.
For $x = 0$ let $y = 0$.

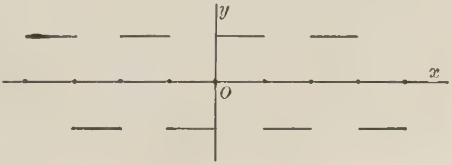


Its graph is indicated in the figure.

An analytic expression of y is

$$y = \lim_{n=\infty} \frac{nx}{1 + nx}.$$

332. For $x = 0, \pm 1, \pm 2, \cdots$ let $y = 0$.
For $n < x < n + 1$, $y = (-1)^n$. *n positive integer or* 0.
For $-(n+1) < x < -n$, $y = -(-1)^n$.



The graph of y is indicated in the figure.

An analytic expression of y is

$$y = \lim_{n=\infty} \frac{(1 + \sin \pi x)^n - 1}{(1 + \sin \pi x)^n + 1}.$$

333. Let $f(x)$, $g(x)$ be two different functions, defined over $\mathfrak{A} = (0, +\infty)$. The inexperienced reader might well believe that we cannot form an analytic expression which represents $f(x)$ in one part of $\mathfrak{A}$, and $g(x)$ for another part. Such an expression is, however, the following:

$$y = \lim_{n=\infty} \frac{x^n f(x) + g(x)}{x^n + 1}.$$

In fact:

$$\text{for } x > 1, \quad y = f(x),$$

$$\text{for } 0 \leq x < 1, \quad y = g(x),$$

$$\text{for } x = 1, \quad y = \tfrac{1}{2}\{f(1) + g(1)\}.$$

Example.

$$f(x) = x^2, \quad g(x) = \cos 2\pi x.$$

Then

$$y = x^2, \text{ for } x \overline{>} 1,$$

$$= \cos 2\pi x, \text{ for } 0 \leq x \leq 1.$$

334. 1. For *rational* x, let $y = a$; for *irrational* x, let $y = b$, where a, b are constants. This function was introduced by *Dirichlet.*

In any little interval, y jumps infinitely often from a to b and back. It seems highly improbable that such a function should admit a simple analytic expression; yet it does.

We have already seen that $sgn\, x$ admits a simple analytic expression.

Consider now

$$y = a + (b - a) \lim_{n=\infty} \operatorname{sgn} (\sin^2 n!\, \pi x). \tag{1}$$

For any rational x, $n!\,x$ finally becomes and remains an integer.

Hence $\sin n!\, \pi x = 0$ for sufficiently large n.

Hence $y = a$ for any rational x.

For an irrational x, $n!\,x$ never becomes an integer. Hence $\sin^2 n!\, \pi x$ lies between 0 and 1, excluding end values.

Therefore

$$\operatorname{sgn} (\sin^2 n!\, \pi x) = 1;$$

and for any irrational x, $y = b$.

Thus 1) is an analytic expression of Dirichlet's function.

The reader should note that it is utterly impossible to intuitionally realize the graph of this function.

2. Similarly, we see that

$$y = f(x) + (g(x) - f(x)) \lim_{n=\infty} \operatorname{sgn}(\sin^2 n! \pi x)$$

equals $f(x)$ when x is rational,

and equals $g(x)$ when x is irrational.

335. A remarkable function is the following. We shall call it *Cauchy's function*, and denote it by $C(x)$, viz.:

$$C(x) = e^{-\frac{1}{x^2}}, \textit{ for } x \neq 0,$$

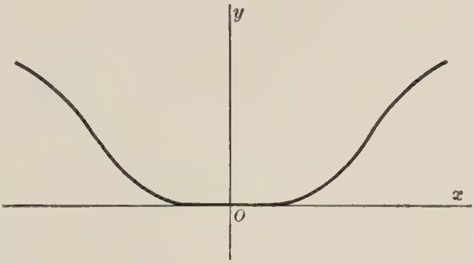
$$= 0, \textit{ for } x = 0.$$

As a limit, we can write it

$$C(x) = \lim_{u=0} e^{-\frac{1}{x^2+u^2}}$$

or

$$C(x) = \lim_{n=\infty} e^{-\frac{1}{x^2+\frac{1}{n}}}.$$



Its graph is given in the figure. Its peculiarity is its remarkable flatness near the origin.

Upper and Lower Limits

336. 1. Let $f(x_1 \cdots x_m) = f(x)$ be defined over D.

Let a be a limiting point of D; a and D may be finite or infinite.

Let $A = a_1, a_2, \cdots$ be a sequence of points in D whose limit is a, such, however, that

$$L = \lim_{n=\infty} f(a_n)$$

exists, finite or infinite. There are an infinity of such sequences.

For all such sequences, let

$$\lambda = \text{Min } L, \qquad \mu = \text{Max } L.$$

These are called respectively the *lower* and *upper limits* of $f(x_1 \cdots x_m)$ at a; we write

$$\lambda = \lim_{x=a} \inf f(x_1 \cdots x_m) = \underline{\lim}_{x=a} f = \underline{\lim} f,$$

$$\mu = \lim_{x=a} \sup f(x_1 \cdots x_m) = \overline{\lim}_{x=a} f = \overline{\lim} f.$$

The lower and upper limits λ, μ may be infinite.

2. When dealing with functions of a single variable, we can have *right* and *left hand upper* and *lower limits*, by considering only values of $x > a$, or $< a$, respectively.

Then

$$R \lim_{x=a} \sup f(x) = \lim_{x=a+0} \sup f(x) = \overline{\lim}_{x=a+0} f(x)$$

$$= R\, \overline{\lim} f = f(\overline{a+0})$$

all denote the right hand upper limit of $f(x)$ at a. A similar notation is employed for the left hand limits.

337. **EXAMPLES***

1.
$$y = \sin \frac{1}{x}.$$

$$\underline{\lim}_{x=0}\, y = -1, \qquad \overline{\lim}_{x=0}\, y = +1.$$

2.
$$y = (1 - x^2) \sin \frac{1}{x}.$$

$$\underline{\lim}_{x=0}\, y = -1, \qquad \overline{\lim}_{x=0}\, y = +1.$$

3.
$$y = (1 + x^2) \sin \frac{1}{x}.$$

$$\underline{\lim}_{x=0}\, y = -1. \qquad \overline{\lim}_{x=0}\, y = +1.$$

4.
$$y = \lim_{n=\infty} \frac{x^n\left(a + \sin \frac{1}{x-1}\right) + b + \sin \frac{1}{x-1}}{1 + x^n}.$$

See 333

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* The reader will do well to roughly sketch the graph of these functions.

We find:

$$y = a + \sin\frac{1}{x-1}, \qquad \textit{for } x > 1.$$

$$y = b + \sin\frac{1}{x-1}, \qquad \textit{for } 0 < x < 1.$$

Hence

$$R\,\overline{\lim_{x=1}}\, y = a + 1; \qquad L\,\overline{\lim_{x=1}}\, y = b + 1;$$

$$R\,\underline{\lim_{x=1}}\, y = a - 1; \qquad L\,\underline{\lim_{x=1}}\, y = b - 1.$$

338. 1. *Let* λ, μ *be the lower and upper limits of* $f(x_1 \cdots x_m)$ *at* $x = a$. *Then there exists for each* $\epsilon > 0$ *a* $\delta > 0$, *such that*

$$\lambda - \epsilon < f(x_1 \cdots x_m) < \mu + \epsilon, \qquad \text{in } V_\delta^*(a),$$

a, finite or infinite.

For, in the contrary case there exist sequences $A = a_1, a_2, \cdots$ such that

$$\lim f(a_n) < \lambda,$$

or

$$\lim f(a_n) > \mu.$$

2. Obviously we have the following:

Let λ, μ *be the lower and upper limits of* $f(x_1 \cdots x_m)$ *at* a. *There exist two sequences* $A = a_1, a_2, \cdots, B = b_1, b_2, \cdots$ *whose limits are* a, *such that*

$$\lim_{n=\infty} f(a_n) = \lambda, \quad \lim_{n=\infty} f(b_n) = \mu.$$

3. Since the maximum and minimum of a variable exist, finite or infinite, we have:

The upper and lower limits of a function always exist finite or infinite. If

$$\underline{\lim_{x=a}}\, f = \overline{\lim_{x=a}}\, f = l,$$

then

$$\lim_{x=a} f = l.$$

CHAPTER VII

CONTINUITY AND DISCONTINUITY OF FUNCTIONS

Definitions and Elementary Theorems

339. 1. Let $f(x_1 \cdots x_m)$ be defined over a domain D. Let $a = (a_1 \cdots a_m)$ be a proper limiting point of D. If

$$\lim_{x=a} f(x_1 \cdots x_m) = f(a_1 \cdots a_m), \qquad (1$$

the function f is continuous at a. In words: *if the limit of f at a is the same as the value of f at a, it is continuous at a.*

The reader should observe that a is not only a limiting point of D, but that it lies in D.

2. The condition 1) may be expressed in the ϵ, δ notation, giving the following *definition of continuity: $f(x_1 \cdots x_m)$ is continuous at a, if for each $\epsilon > 0$ there exists a $\delta > 0$, such that*

$$|f(x_1 \cdots x_m) - f(a_1 \cdots a_m)| < \epsilon, \qquad \textit{in } V_\delta(a).$$

3. A function which is continuous at all the proper limiting points of D is said to be *continuous in D.* We suppose that D has at least *one* proper limiting point.

4. Consider the function

$$f(x, y) = \frac{xy}{x^2 + y^2}, \qquad \textit{at points different from the origin.}$$

$$= 0, \qquad \textit{at the origin.}$$

We saw, 319, that

$$\lim_{x=0,\, y=0} f(x, y)$$

does not exist. Thus f is not continuous at the origin.

At the same time f considered as a function of x alone, or considered as a function of y alone, is continuous.

This example illustrates, therefore, the fact that because $f(x_1 \cdots x_m)$ is a continuous function of each variable separately, we cannot, therefore, assert that f considered as a function of $x_1 \cdots x_m$ is continuous.

340. The following theorems will be found useful in determining whether $f(x_1 \cdots x_m)$ is continuous at a or not.

From 277, 1 and 317, 1 we have at once:

Let $f(x_1 \cdots x_m)$, $g(x_1 \cdots x_m)$ *be continuous at* a. *Then*

$$f \pm g, \qquad f \cdot g,$$

$$\frac{f}{g}, \qquad g(a_1 \cdots a_m) \neq 0$$

are continuous at a.

341. From 292 and 317, 2 we have at once:

Let
$$u_1 = \phi_1(x_1 \cdots x_n) \quad \cdots \quad u_m = \phi_m(x_1 \cdots x_n)$$

be continuous at $x = a = (a_1 \cdots a_n)$. *At* $x = a$, let

$$u_1 = b_1 \quad \cdots \quad u_m = b_m.$$

Let
$$y = f(u_1 \cdots u_m)$$

be continuous at $u = b = (b_1 \cdots b_m)$. *Then* y *considered as a function of the* x'*s is continuous at* $x = a$.

In a less explicit form, we may state this theorem:

A continuous function of a continuous function is a continuous function.

342. *In order that* $f(x_1 \cdots x_m)$ *be continuous at* a, *it is necessary and sufficient that for each* $\epsilon > 0$ *exists an undeleted vicinity* $V(a)$, *such that for any two points* x', x'' *in it,*

$$|f(x') - f(x'')| < \epsilon.$$

This follows at once from 284 and 317, 1.

Continuity of the Elementary Functions

343. *The integral rational functions are everywhere continuous.*

Let
$$y = x^n. \qquad n \text{ positive integer.}$$

Then, by 299, y is continuous at every point x in $\mathfrak{R}$. Hence, by 340,
$$a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$$
is everywhere continuous. Thus the theorem is proved for one variable.

Let
$$y = \Sigma A x_1^{s_1} x_2^{s_2} \cdots x_m^{s_m},$$
where the s's are positive integers or 0.

Each term of y, viz.
$$t = A x_1^{s_1} \cdots x_m^{s_m}, \tag{1}$$
is continuous at an arbitrary point x. For,

let
$$u_1 = x_1^{s_1}, \quad \cdots \quad u_m = x_m^{s_m}.$$

Then the term t becomes the product
$$u_1 u_2 \cdots u_m,$$
which, considered as a function of the u's, is continuous, by 340. On the other hand, each u_κ is a continuous function of the x's. Hence, by 341, t, considered as a function of the x's, is continuous at every point x in $\mathfrak{R}_m$. Hence y, being a sum of the terms t, is continuous, by 340.

344. *The rational functions are continuous everywhere in their domain of definition D.*

We saw, in 228, that the domain D of
$$y = \frac{\Sigma A x_1^{r_1} \cdots x_m^{r_m}}{\Sigma B x_1^{s_1} \cdots x_m^{s_m}} = \frac{F}{G}$$
embraces all points of $\mathfrak{R}_m$, except the zeros of the denominator, which are the poles of y.

By 343, F and G are everywhere continuous; hence, by 340, y is everywhere continuous, except at the poles of y.

345. 1. *The circular functions are continuous at every point of their domain of definition.*

From 202, we saw that $\sin x$, $\cos x$, are defined for all points of $\mathfrak{R}$; while

$$\tan x = \frac{\sin x}{\cos x}, \qquad \cot x = \frac{\cos x}{\sin x},$$

$$\sec x = \frac{1}{\cos x}, \qquad \operatorname{cosec} x = \frac{1}{\sin x},$$

are defined for all points of $\mathfrak{R}$, except for the zeros of the denominators in the above equations.

From 296, $\sin x$ and $\cos x$ are everywhere continuous. The rest of the theorem follows now from 340.

2. *The one-valued functions*

$$\operatorname{arc} \sin x, \quad \operatorname{arc} \cos x, \quad \operatorname{arc} \operatorname{tg} x, \quad \operatorname{arc} \operatorname{ctg} x$$

are continuous at every point of their domains of definition.

This follows at once from 294.

346. 1. *The exponential functions are everywhere continuous.*

This follows at once from 298, 5.

2. *The logarithmic functions are everywhere continuous in their domain of definition.*

Let

$$y = \log_b x.$$

Then

$$y = \frac{\log_e x}{\log_e b}.$$

Hence y is continuous at a point x, if $\log_e x$ is. But this is continuous for every $x > 0$, by 300.

The demonstration also may be given by 294.

Discontinuity

347. If

$$\lim_{x=a} f(x_1 \cdots x_m)$$

does not exist; or if it exists, and is different from $f(a)$, should f be defined at a, we say f is *discontinuous at* a, and a is a *point of discontinuity of* f.

Discontinuities are of two kinds:

Finite discontinuities, when f is limited in $V^*(a)$.

Infinite discontinuities, when f is unlimited in every $V^*(a)$.

348. We consider now in detail some of the ways in which a function of a single variable $f(x)$ may be discontinuous at a point a.

Finite Discontinuities

1.
$$f(a+0)=f(a-0)\neq f(a).$$

Such a discontinuity is called a *removable discontinuity*.

Such a function is

$$y=\lim_{n=\infty}\frac{nx}{1+nx},$$

considered in 331.

2. $f(a+0), f(a-0)$ exist, but are different.

Such a function is

$$y=\operatorname{sgn} x,$$

considered in 330.

3. If $f(x)$ is defined at a, and $f(a)=f(a+0)$, we say f is *continuous on the right, at a.*

If $f(a)=f(a-0)$, f is *continuous on the left, at a.*

4. Either $f(a+0)$, or $f(a-0)$, or both do not exist.

Such a function is

$$y=\sin\frac{1}{x}.$$

We considered this function in 256. Here neither $f(0+0)$ nor $f(0-0)$ exist. Also f is not defined for $x=0$.

Infinite Discontinuities

349. 1. As x approaches a from either side, $f(x)$, either monotone increases or monotone decreases.

Ex. 1.

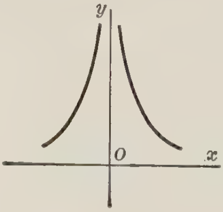
$$y=\frac{1}{x^2},$$

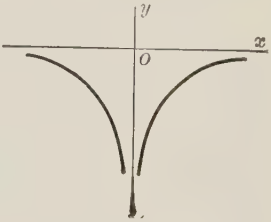
at 0.

Ex. 2.

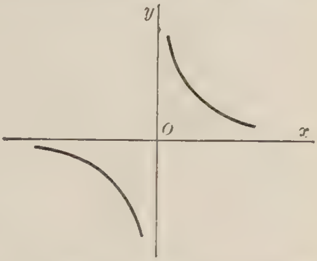
$$y=\frac{-1}{|x|},$$

at 0.





2. As x approaches a, $f(x)$ increases monotone on one side, and decreases monotone on the other.



Ex. 3. $y = \frac{1}{x}$, at 0.

Ex. 4. $y = \tan x$, at $\frac{\pi}{2}$.

3. As x approaches a, y oscillates infinitely often about a base curve, belonging to the types defined in 1 or 2. The amplitude of the oscillations is limited.

Ex. 5. $y = \frac{1}{x^2} + x \sin \frac{1}{x}$, at $x = 0$.

Here y oscillates about the base curve

$$y = \frac{1}{x^2};$$

and the amplitude of the oscillations converges to 0 as x approaches 0.

Ex. 6. $y = \frac{1}{x} + \sin \frac{1}{x}$, at $x = 0$.

Here y oscillates about

$$y = \frac{1}{x}; \quad (1$$

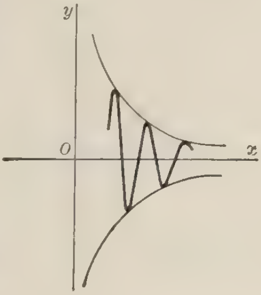
and the amplitude of the oscillations remains the same, viz. ±1 above the curve 1).

4. The discontinuities considered in the preceding three cases are such that either

$$\lim_{x=a} y$$

is infinite, or at least the right and left hand limits at a are infinite and of opposite signs. Such points of discontinuities of $f(x)$ are called *infinities;* we also say *$f(x)$ is infinite* at such points.

5. In either or both the right and left hand vicinities of a, y is unlimited, while the corresponding (infinite) limits do not exist.



Ex. 7. $y = \frac{1}{x} \sin \frac{1}{x}$. $x = 0$.

Here y oscillates between the two hyperbolas

$$y = \pm \frac{1}{x}.$$

The amplitude of the oscillations increases indefinitely as x approaches 0.

Ex. 8.
$$y = \frac{1}{x} + \frac{1}{x} \sin \frac{1}{x}. \qquad x = 0.$$

Here y oscillates about the base curve

$$y = \frac{1}{x}.$$

The amplitude of the oscillations increases indefinitely as x approaches 0.

Ex. 9.
$$y = e^{\frac{1}{x}}.$$

Here
$$L \lim_{x=0} y = 0 ; \quad R \lim_{x=0} y = +\infty.$$

Ex. 10.
$$y = e^{\frac{1}{x}} \sin \frac{1}{x}.$$

Here
$$L \lim_{x=0} y = 0 ; \quad R \lim_{x=0} y \text{ does not exist.}$$

Some Properties of Continuous Functions

350. 1. *If $f(x_1 \cdots x_m)$ is continuous in a limited perfect domain D, it is limited in D.*

For if f were not limited,

$$\text{Max} |f| = +\infty. \qquad (1$$

Then, by 269, there is a point a of D in whose vicinity 1) holds. This is impossible. For, since f is continuous,

$$f(a_1 \cdots a_m) - \epsilon < f(x_1 \cdots x_m) < f(a_1 \cdots a_n) + \epsilon$$

in $V(a)$.

2. The theorem 1 does not need to be true if D is not limited.

EXAMPLE. $D = (0, +\infty)$, $f(x) = x^2$.

Here f is continuous in D, but f is not limited.

3. The theorem 1 does not need to hold if D is not perfect.

EXAMPLE. $D = (0^*, 1)$, $f(x) = \frac{1}{x}$.

Here f is continuous in D, but f is not limited.

351. 1. *At $x = a$ let $f(x_1 \cdots x_m)$ be continuous and $\neq 0$. Then in $V(a)$,*

$$\text{sgn}\, f(x_1 \cdots x_m) = \text{sgn}\, f(a_1 \cdots a_m).$$

For, since f is continuous at a, we have

$$\epsilon > 0, \quad \delta > 0, \quad |f(x) - f(a)| < \epsilon, \qquad V_\delta(a).$$

Hence

$$f(a) - \epsilon < f(x) < f(a) + \epsilon.$$

Since ϵ is arbitrarily small, we can take it so small that

$$f(a), \quad f(a) - \epsilon, \quad f(a) + \epsilon$$

all have the same sign.

2. The theorem 1 gives us:

At $x = a$ let $f(x_1 \cdots x_m)$ be continuous and $\neq 0$. Then there exists a $\rho > 0$, such that

$$|f| > \rho, \qquad \text{in } V(a).$$

352. Let $f(x_1 \cdots x_m)$ be defined over a domain D. By definition, it is continuous in D when, for each proper limiting point x in D,

$$\lim_{h=0} f(x_1 + h_1 \cdots x_m + h_m) = f(x_1 \cdots x_m),$$

the points $x + h$ lying in D.

If $f(x_1 + h_1 \cdots x_m + h_m)$ not only converges to $f(x_1 \cdots x_m)$ in D, but *converges uniformly*, we say f is *uniformly continuous* in D.

We have now the very important theorem:

If $f(x_1 \cdots x_m)$ is continuous in a limited perfect domain D, it is uniformly continuous in D.

Making use of the notation of 325, we have only to show that

$$\delta_0 = \text{Min } \delta, \qquad \text{for } D$$

is > 0.

Suppose it were not, *i.e.* let $\delta_0 = 0$. We show that this assumption leads to a contradiction.

For, by 269, there is a point a in D, such that in $V(a)$

$$\text{Min } \delta = \delta_0 = 0. \tag*{(1}$$

This is impossible. In fact, by 342, there exists for each $\epsilon > 0$, a δ', such that for any pair of points x', x'' in $V_{\delta'}(a)$

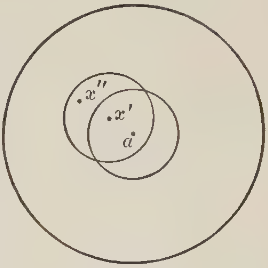
$$|f(x') - f(x'')| < \epsilon. \tag*{(2}$$

Let $\delta'' < \frac{1}{2}\delta'$. Let now x' be any point in $V_{\delta''}(a)$.

Then every point x'' in $V_{\delta''}(x')$ falls in $V_{\delta'}(a)$, by 249.

Thus, for any such pair of points x', x'', 2) holds.

Thus, for no point x' in $V_{\delta''}(a)$ does δ sink below δ'', and this contradicts 1).



353. *Let $f(x_1 \cdots x_m)$ be continuous in the limited perfect domain D. For each $\epsilon > 0$ there exists a cubical division of D, of norm $\delta > 0$, such that*

$$|f(x') - f(x'')| < \epsilon \tag{1}$$

for any pair of points x', x'' in any one of the cells Δ into which D falls.

For, since f is uniformly continuous in D, let $\sigma > 0$ be such that 1) holds for any point x'' of $V_\sigma(x')$. Let now the norm of the cubical division be

$$\delta < \frac{\sigma}{\sqrt{m}}.$$

Suppose x', x'' were a pair of points in some cell Δ, such that 1) does not hold. Since

$$\text{Dist}(x', x'') \leq \delta\sqrt{m} < \sigma, \text{ by 244, 8, 9,}$$

x'' lies in $V_\sigma(x')$. But then 1) holds for x', x''. We are thus led to a contradiction.

354. 1. *Let $f(x_1 \cdots x_m)$ be continuous in the perfect limited domain D. Then f takes on its extreme values in D.*

Before giving the demonstration, let us illustrate the content of this theorem. Let

$$D = (0^*, 1^*), \text{ and } y = x^2.$$

Then, for D,

$$\text{Max } y = +1, \quad \text{Min } y = 0.$$

But y does not take on either of these extremes in D. This is due to the fact that D is not perfect.

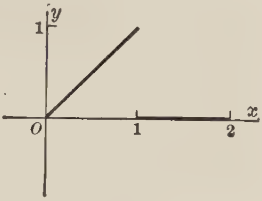
2. Let
$$D=(0, +\infty), \text{ and } y=\frac{1}{1+x}.$$
$$\text{Max } y=1, \quad \text{Min } y=0.$$

Thus y takes on its maximum, viz. at $x=0$, but does not take on its minimum. This is due to the fact that D is not limited.

3. Let
$$D=(0, 2),$$
and
$$y=\lim_{n=\infty}\frac{x}{x^n+1}.$$

This function is a particular case of that in 333.

For
$$0 \leq x < 1, \quad y=x;$$
for
$$x=1, \quad y=\tfrac{1}{2};$$
for
$$1<x\leq 2, \quad y=0.$$
Hence
$$\text{Min } y=0, \quad \text{Max}=1.$$



The function takes on its minimum value in D, but *not* its maximum. This is due to the discontinuity of y at 1.

355. 1. We give now the demonstration of 354.

Let e be an extreme of $f(x_1 \cdots x_m)$. Then, by 269, there is a point a in D such that e is an extreme of f in every $V(a)$.

Thus, taking $\epsilon>0$ small at pleasure, there is at least one point x' in any $V(a)$, such that
$$|f(x')-e|<\frac{\epsilon}{2}. \tag{1}$$

Since f is continuous at a, there is a $\delta>0$, such that for any x in $V_\delta(a)$
$$|f(x)-f(a)|<\frac{\epsilon}{2}. \tag{2}$$

In 2) set $x=x'$, and add to 1); we get
$$|f(a)-e|<\epsilon.$$

Hence, by 87, 5,
$$f(a)=e.$$

2. As corollary we have:

Let $f(x_1 \cdots x_m)$ be continuous and >0 in the perfect limited domain D. Then
$$\text{Min} f>0, \qquad \textit{in } D.$$

356. *In the interval* $\mathfrak{A} = (a, b)$ *let* $f(x)$ *be continuous. Let it have opposite signs at* a *and* b. *Then* f *vanishes for some point* c *within* $\mathfrak{A}$.

Let us form a partition (A, B) with the points of $\mathfrak{A}$. The class A is formed thus. Not only shall

$$\operatorname{sgn} f(x) = \operatorname{sgn} f(a) \tag{1}$$

at every point of A, but between a and any point of A shall 1) hold. In B we throw the other points of $\mathfrak{A}$.

Let c generate this partition. Then in any $V(c)$, f has opposite signs. But if $f(c)$ were $\neq 0$, by 351, we could take δ so small that $f(x)$ has only one sign in V_δ. This leads to a contradiction. Hence $f(c) = 0$.

The point c cannot be an end point of $\mathfrak{A}$, for at these points $f \neq 0$ by hypothesis.

357. *Let* $f(x)$ *be continuous in* $\mathfrak{A} = (a, b)$. *Let* $\operatorname{Min} f(x) = \alpha$, $\operatorname{Max} f(x) = \beta$ *in* $\mathfrak{A}$. *Then* $f(x)$ *takes on every value in* (α, β) *at least once, while* x *passes from* a *to* b.

By 354, $f(x)$ takes on its extreme values in $\mathfrak{A}$. Let, therefore,

$$f(x') = \alpha, \quad f(x'') = \beta.$$

To fix the ideas, let $0 < \alpha < \beta$.

Let $\alpha < \gamma < \beta$. Set

$$g(x) = f(x) - \gamma.$$

Then

$$g(x') < 0,$$

$$g(x'') > 0.$$

Hence, by 356, g vanishes at some point in (x', x''). At this point $f(x) = \gamma$.

358. *Let* $y = f(x)$ *be a continuous univariant function in the interval* (a, b). *Let* $\alpha = f(a)$, $\beta = f(b)$. *Then the inverse function* $x = g(y)$ *is a one-valued univariant continuous function in* (α, β).

By 214, $g(y)$ is a one-valued univariant function in its domain of definition E. By 357, $E = (\alpha, \beta)$. By 294, $g(y)$ is continuous in (α, β).

The Branches of Many-valued Functions

359. Let $F(x_1 \cdots x_m)$ be a many-valued function in D. We can form a one-valued function $f(x_1 \cdots x_m)$ over a domain $\Delta \leq D$ by assigning to f at each point of Δ *one* of the values of F at this point, according to some law.

A common way to do this is to assign to f such values that it is *continuous* in Δ. In this case we say $f(x_1 \cdots x_m)$ is a *branch* of the many-valued function F.

A point, at which two or more branches meet, may be called a *branch point*.

360. Ex. 1. The equation

$$y^2 = x \qquad (1$$

defines a two-valued function of x in the interval $(0, +\infty)$.

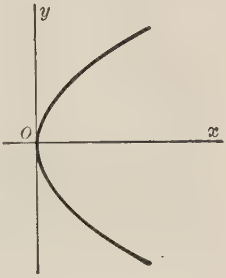
One branch is the one-valued function

$$\sqrt{x}\,; \qquad (2$$

the other branch is

$$-\sqrt{x}. \qquad (3$$

The graph of 2) embraces the points in the upper half of the parabola 1); the graph of 3) is the lower half of this parabola.



361. Ex. 2. The equation

$$x^4 - ax^2y + by^3 = 0$$

defines a three-valued function of x whose graph is given in Fig. 1.

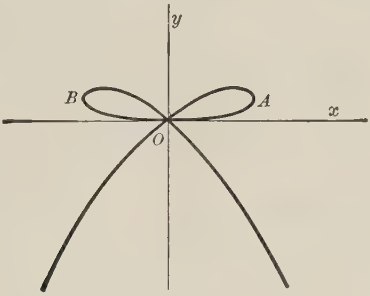


FIG. 1.

Still preserving the continuity, we can define the branches of y in several ways, according to the path we take on leaving O.

In any case, however, we must stop at the points A, B, at which the ordinate is tangent to the curve. For, if we passed beyond these points, we would no longer have a one-valued function.

In Figs. 2, 3, 4 we illustrate various ways of choosing a branch.

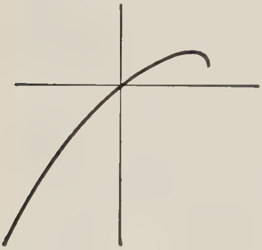


FIG. 2.

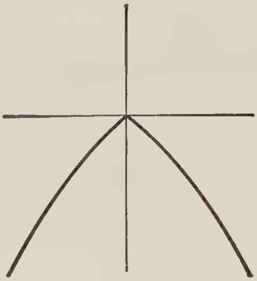


FIG. 3.

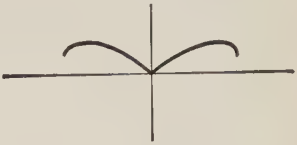


FIG. 4.

Notion of a Curve

362. 1. In elementary mathematics one meets with a great variety of curves. Their equations may be expressed, confining ourselves for the moment to the plane, in one of the three forms

$$y = f(x), \tag*{(1}$$

$$x = \phi(u), \ y = \psi(u), \tag*{(2}$$

$$F(xy) = 0. \tag*{(3}$$

When the curve is given by 1) or 2), it is said to be defined *explicitly;* when given by 3), it is said to be defined *implicitly.*

We observe that 1) is a special case of 2). For we have only to write

$$x = u, \ y = \psi(u),$$

to reduce 1) to 2).

When the curve is given by 2), it is said to be expressed in *parametric form.*

We note that 1) is also a special case of 3). For we have only to set

$$F(xy) = y - f(x),$$

to bring 1) to the form 3).

2. It is customarily thought that the *notion of a curve* is a very simple one; but we shall see that this is not so. On the contrary, it is a very obscure and complex notion. Reserving the discussion of the notion of a curve in general until later, it is well to give a preliminary definition.

Let $\phi(u)$, $\psi(u)$ be continuous one-valued functions of u for an interval $\mathfrak{A} = (a, b)$, finite or infinite.

Set

$$x = \phi(u), \quad y = \psi(u).$$

Let u range over $\mathfrak{A}$. The points P whose coördinates are x, y will form a point aggregate which we call a *curve*. The point x, y is the image of the point u.

Ex. 1. Let $\phi(u) = u$, $\psi(u) = u^2$. The curve so defined is a parabola whose axis is the y-axis.

Ex. 2. Let $\phi(u) = a \cos u$, $\psi(u) = b \sin u$. The curve so defined is an ellipse.

Still more generally an aggregate of a finite number of curves may be called a curve C.

Each one of the individual curves which enter in C may be called an *arc* or *part* of C.

If a curve or a piece of a curve is such that y is a one-valued monotone function of x, we shall say the curve or the piece of it is *monotone*. In the same way we shall extend the terms *monotone*, *increasing*, *univariant*, etc., to curves or arcs of curves.

3. Let u range from a to b. If to two different points u', u'' of $\mathfrak{A} = (a, b)$ corresponds the same point $P(x, y)$, this point P will be called a *multiple point* of C.

Let the coördinates xy take on the same pair of values at the end points of $\mathfrak{A} = (a, b)$. Then C is called a *closed curve*. If C has no multiple points in (a, b^*), we shall say C is a *closed curve without multiple points*.

4. The extension of these notions to n-dimensional space, by setting

$$x_1 = \phi_1(u), \cdots x_n = \phi_n(u),$$

is too obvious to need comment.

CHAPTER VIII

DIFFERENTIATION

FUNCTIONS OF ONE VARIABLE

Definitions

363. Let $y = f(x)$ be defined over a domain D for which a is a proper limiting point. The quotient

$$\frac{\Delta y}{\Delta x} = \frac{f(x) - f(a)}{x - a}, \qquad x \textit{ in } D, \tag{1}$$

is called the *difference quotient* at a.

If we set $x = a + h$, we have also

$$\frac{\Delta y}{\Delta x} = \frac{f(a+h) - f(a)}{h}. \tag{2}$$

Let

$$\lim_{\Delta x = 0} \frac{\Delta y}{\Delta x} = \eta \tag{3}$$

exist, finite or infinite. Then η is called *the differential coefficient* of $f(x)$ at a, and is denoted by

$$f'(a).$$

Let Δ be the aggregate of points in D for which η is finite or infinite. The corresponding values of η define a function of x, called the *derivative* of $f(x)$, more specifically, the *first derivative* of $f(x)$. It is represented variously by

$$f'(x), \quad D_x f(x), \quad \frac{df}{dx}, \quad \frac{dy}{dx}. \tag{4}$$

The function $f'(x)$ is said to be obtained from $f(x)$ by the process of *differentiation*. A function which admits a derivative is said to be *differentiable*.

Since $f'(x)$ may be infinite, the reader will observe that its values lie in $\overline{\mathfrak{R}}$. Cf. 276.

364. In the same way, the right and left hand limits at $x=a$,

$$R\lim\frac{\Delta y}{\Delta x},\qquad L\lim\frac{\Delta y}{\Delta x},$$

give rise to *right* and *left hand differential coefficients at a*. These we denote by

$$Rf'(a),\quad Lf'(a).$$

These in turn give rise to *right* and *left hand derivatives*, which we may denote, prefixing R and L before the symbols, 4) in 363.

When speaking of differential coefficients and derivatives in the future, we shall mean those defined in 363, unless the contrary is expressly stated.

However, much that we prove for $f'(a)$ and $f'(x)$ may be applied at once to the corresponding unilateral differential coefficients and derivatives.

Geometric Interpretations

365. Let P and R be the points on the graph of

$$y=f(x),$$

corresponding to $x=a$ and $x=a+\Delta x$, Fig. 1.

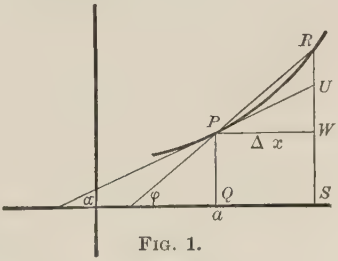


FIG. 1.

Then

$$PW=\Delta x=h,\quad RW=\Delta y.$$

If the secant PR makes the angle ϕ with the x-axis,

$$\frac{\Delta y}{\Delta x}=\frac{RW}{PW}=\tan\phi.$$

That is: *the difference quotient is the tangent of the angle that the secant makes with the x-axis.*

Suppose now y is continuous in a little interval about $x=a$; if the secant PR approaches a limiting position PU, as R approaches the fixed point P from either side, we say PU is the *tangent* to the curve at P.

Evidently, if $f'(a)$ is finite,

$$f'(a)=\lim\frac{\Delta y}{\Delta x}=\lim\tan\phi=\tan\alpha,$$

where α is the angle that the tangent line makes with the x-axis.

If $f'(a)=\pm\infty$, the tangent line is parallel to the y-axis.

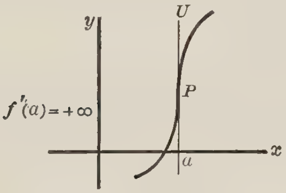


Fig. 2.

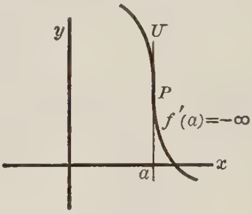


Fig. 3.

Such cases are shown in Figs. 2 and 3.

The point P is a *point of inflection with vertical tangent.*

For an example of such a function, see 388, 5.

366. 1. When the differential coefficient at a does not exist, finite or infinite, the right and left differential coefficients may. They are then different.

If both are finite, we have a case illustrated by Fig. 1.

Such a function is

$$f(x)=x\frac{e^{\frac{1}{x}}-1}{e^{\frac{1}{x}}+1},\qquad\text{for } x\neq 0;$$

$$=0,\qquad\text{for } x=0.$$

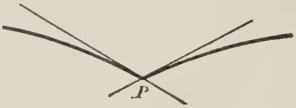


Fig. 1.

Here

$$Rf'(0)=+1,\quad Lf'(0)=-1.$$

If one is finite and the other infinite, we have a case illustrated by Fig. 2.

The points P in Figs. 1, 2 are called *angular points.*

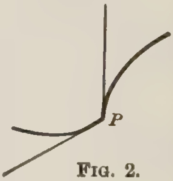


Fig. 2.

2. When both differential coefficients are infinite, but of opposite signs, we have a case illustrated by Figs. 3, 4.

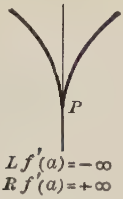


FIG. 3.

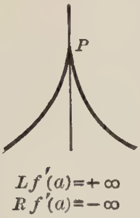


FIG. 4.

Here P is a *cusp with vertical tangent.*

See 388, 3 for an example of such a function.

3. In Case 1 the curve has *not one* but *two* tangents at P; viz. a *right* and a *left hand tangent.* Case 2 may be considered as a special or limiting case of 1. The curve has a tangent at P.

In both cases the direction of motion along the curve changes abruptly.

When we say "a curve has at every point a tangent," we exclude Case 1.

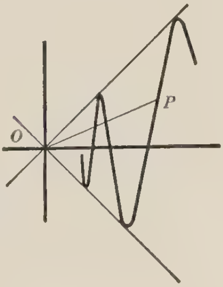
Non-existence of the Differential Coefficient

367. 1. We consider now some examples of *continuous* functions for which the differential coefficient on either side of certain points does not exist.

Let

$$y = f(x) = x \sin \frac{\pi}{x}, \qquad \text{for } x \neq 0;$$

$$= 0, \qquad \text{for } x = 0.$$



The graph Γ of y is given in the adjoining figure.

Evidently Γ oscillates between the two lines

$$y = \pm x, \qquad (1$$

with increasing rapidity as x approaches 0.

For $x \neq 0$, y is evidently continuous.

For $x = 0$, y is also continuous, since

$$\lim_{x=0} x \sin \frac{\pi}{x} = 0.$$

At the origin the secant line OP oscillates between the two lines 1), and obviously does not approach any fixed position as P approaches 0 from either side. Thus Γ has no tangent at all at O.

This result is verified at once analytically.

For,

$$\frac{\Delta y}{\Delta x} = \sin \frac{\pi}{\Delta x}, \qquad \text{at } x = 0;$$

and as $\Delta x \doteq 0$, $\sin \dfrac{\pi}{\Delta x}$ oscillates infinitely often between ± 1.

2. For use later, let us find $\dfrac{dy}{dx}$ for $x = \dfrac{1}{n}$.

We have, setting $\Delta x = h$,

$$\frac{\Delta y}{\Delta x} = \frac{1}{h}\left(\frac{1}{n} + h\right) \sin \frac{\pi}{\dfrac{1}{n} + h}$$

$$= \frac{1 + nh}{nh} \sin \frac{n\pi}{1 + nh}.$$

But

$$\sin \frac{n\pi}{1 + nh} = \sin\left(n\pi - \frac{n^2\pi h}{1 + nh}\right) = -(-1)^n \sin \frac{n^2\pi h}{1 + nh}.$$

Hence, setting

$$u = \frac{n^2\pi h}{1 + nh},$$

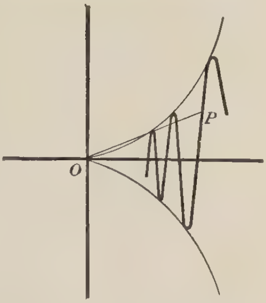
$$\frac{\Delta y}{\Delta x} = -(-1)^n n\pi \frac{\sin u}{u};$$

and thus

$$\lim_{\Delta x=0} \frac{\Delta y}{\Delta x} = -(-1)^n n\pi \lim_{u=0} \frac{\sin u}{u}$$

$$= -(-1)^n n\pi, \text{ by 301.}$$

368. Let
$$y = f(x) = x^2 \sin\frac{\pi}{x}, \quad x \neq 0;$$
$$= 0, \qquad x = 0.$$



Evidently y is everywhere continuous even at 0.

The graph Γ of y oscillates between the two parabolas
$$y = \pm x^2$$
with increasing rapidity as x approaches 0.

As P approaches 0, the secant OP oscillates between narrower and narrower limits, which limits converge on both sides toward the x-axis. Evidently,
$$f'(0) = \lim \frac{\Delta y}{\Delta x} = 0;$$
and Γ has a tangent at 0, viz. the axis of x.

This result is verified analytically at once.

For,
$$\frac{\Delta y}{\Delta x} = \Delta x \sin\frac{\pi}{\Delta x} \text{ at } 0,$$
and
$$\lim_{\Delta x = 0} \Delta x \sin\frac{\pi}{\Delta x} = 0.$$

369. Let $A = 0, \pm 1, \pm \frac{1}{2}, \pm \frac{1}{3}, \cdots$

For x not in A, let $y = f(x) = x \sin\frac{\pi}{x} \sin\frac{\pi}{\sin\frac{\pi}{x}}$.

For x in A, let $y = 0$.

Here y is everywhere continuous, even at the points of A.

Let C be the graph of y, and Γ the graph of
$$y_1 = x \sin\frac{\pi}{x},$$
considered in 367.

In Fig. 1, the full curve represents an arc of Γ for an interval $I_n = (a_n, b_n)$, $a_n = \frac{1}{n}$, $b_n = \frac{1}{n-1}$. The dotted curve, call it Γ', is symmetrical to Γ.

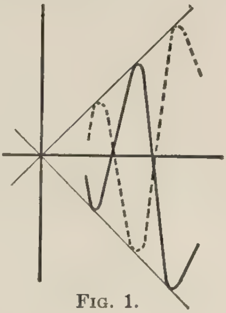


FIG. 1.

We observe now that y is obtained by multiplying the ordinate y_1 of Γ by the factor

$$y_2 = \sin \frac{\pi}{\sin \frac{\pi}{x}}.$$

As x approaches an end point of I_n,

$$\sin \frac{\pi}{x} \doteq 0.$$

Hence y_2 oscillates infinitely often between ± 1. The effect of the factor y_2 in $y = y_1 y_2$ is thus to bend Γ in I_n an infinite number of times, so that the resulting curve, a portion of C, lies between Γ and Γ'.

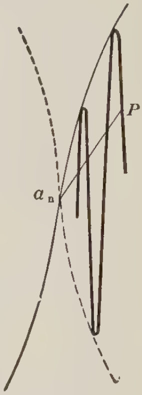


FIG. 2.

This is represented in Fig. 2, where the light and dotted curves are Γ and Γ', and the heavy curve is C.

At one of the points of A, as a_n, the secant a_nP oscillates with increasing rapidity as P approaches a_n from either side.

Since

$$\frac{dy_1}{dx} = -(-1)^n n\pi, \text{ by 367, 2,}$$

the tangents to Γ and Γ' are not the x-axis. Hence the limits of oscillation of the secant do not converge to 0, and hence the secant a_nP does not converge to some fixed position as x approaches a_n.

Thus y has no differential coefficient at any point of A, and its graph C has at these points no tangent.

Since 0 is the limiting point of A, there are an infinity of these singular points in the vicinity of the origin.

370. Let $A = 0, \pm 1, \pm 2, \cdots$

For x not in A, let $y = f(x) = x^2 \sin \frac{\pi}{x} \sin \frac{\pi}{\sin \frac{\pi}{x}}$.

For x in A, let $y = 0$.

The reasoning of 369 may be applied here. The graph of y oscillates between the two curves

$$y_1 = \pm x^2 \sin \frac{\pi}{x},$$

discussed in 368.

There is no tangent at the points $\pm 1, \pm 2, \cdots$ while at the origin there is a tangent, viz. the x-axis.

The graph C of y presents therefore this peculiarity: in the vicinity of the origin there are an infinity of points at which C has no tangents; yet at the origin itself C has a tangent.

371. In 369 and 370, the aggregate A is of the first order, by 263, 2.

It is easy by the *process of iteration* to form continuous functions which have no differential coefficient over an aggregate A, of order m.

Let
$$\theta(x) = \sin \frac{\pi}{x}$$
and
$$y = x\theta(x)\theta^{(2)}(x) \cdots \theta^{(m+1)}(x).$$

This expression does not define y at points involving division by zero. At these points, call their aggregate A, we set $y = 0$. It is easy to show that y is everywhere continuous and that it has neither right nor left hand differential coefficients at any point of A. The aggregate is of order m. See 259, 260.

Fundamental Formulæ of Differentiation

372. As many American and English works on the calculus derive these formulæ in an incorrect or incomplete manner, we shall deduce some of them here. We shall, at the same time, prove them under conditions slightly more general than usual. As domain of definition D of our functions $y, u, v, \cdots$ we take any aggregate having proper limiting points. The domain of definition Δ of their derivatives will embrace, at most, the proper limiting points of D.

It is convenient to represent
$$y(x+h),\ u(x+h),\ \cdots \text{ by } \bar{y},\ \bar{u},\ \cdots \text{ etc.,}$$
and
$$\frac{dy}{dx},\ \frac{du}{dx},\ \cdots \text{ by } y',\ u',\ \cdots \text{ etc.}$$

373. We begin by proving:

If the differential coefficient $f'(a)$ is finite, $f(x)$ is continuous at a.

For, since

$$\lim_{h=0} \frac{f(a+h)-f(a)}{h} = f'(a),$$

we have, for each $\epsilon > 0$, a $\delta > 0$, such that, if $|h| < \delta$,

$$\frac{f(a+h)-f(a)}{h} = f'(a) + \epsilon'. \qquad |\epsilon'| < \epsilon.$$

Hence

$$f(a+h) = f(a) + h(f'(a) + \epsilon').$$

Therefore,

$$\lim_{h=0} f(a+h) = f(a),$$

which states that f is continuous at a.

374. *If y is constant in D, $y' = 0$.*

For

$$\frac{\Delta y}{\Delta x} = 0,$$

for any point of D.

375. *Let $y = u \pm v$. Let u', v' be finite in Δ. Then $y' = u' \pm v'$ in Δ.*

For

$$\frac{\Delta y}{\Delta x} = \frac{\Delta u}{\Delta x} \pm \frac{\Delta v}{\Delta x}. \qquad (1$$

Since u', v' exist and are finite, we can apply 277, 2, to 1).

376. *Let $y = uv$. Let u', v' be finite in Δ.*

Then, in Δ,

$$y' = uv' + vu'. \qquad (1$$

For

$$\frac{\Delta y}{\Delta x} = \frac{\overline{uv} - uv}{\Delta x} = \frac{(u+\Delta u)(v+\Delta v) - uv}{\Delta x}$$

$$= \frac{\bar{u}\Delta v + v\Delta u}{\Delta x} = \bar{u}\frac{\Delta v}{\Delta x} + v\frac{\Delta u}{\Delta x}. \qquad (2$$

By 373,

$$\lim \bar{u} = u.$$

By hypothesis,

$$\lim \frac{\Delta u}{\Delta x} = u', \ \lim \frac{\Delta v}{\Delta x} = v'.$$

Hence, passing to the limit in 2), we get 1).

377. 1. *Let* $y = \frac{u}{v}$. *Let* u', v' *be finite and* $v \neq 0$, *in* Δ. *Then*

$$y' = \frac{vu' - uv'}{v^2}, \ \textit{in } \Delta. \tag{1}$$

For

$$\frac{\Delta y}{\Delta x} = \frac{v\Delta u - u\Delta v}{v\bar{v}\Delta x} = \frac{1}{\bar{v}} \frac{\Delta u}{\Delta x} - \frac{u}{v} \cdot \frac{1}{\bar{v}} \frac{\Delta v}{\Delta x}. \tag{2}$$

By 373,

$$\lim \bar{v} = v.$$

By hypothesis,

$$\lim \frac{\Delta u}{\Delta x} = u', \ \lim \frac{\Delta v}{\Delta x} = v'.$$

Passing now to the limit in 2), we get 1).

We observe, by 351, that $\bar{v} \neq 0$ for Δx sufficiently small, since v is continuous and $\neq 0$ at x. It is therefore permissible to divide by $\bar{v}$, as in 2).

2. *Criticism.* Some writers derive 1) as follows. From

$$y = \frac{u}{v}$$

they get

$$yv = u.$$

They now apply 376, which gives

$$u' = yv' + vy', \tag{3}$$

which, solved, gives 1).

This method is incorrect. For to get 3), by using 376, we must impose the condition that y' exists and is finite. But nothing in this form of demonstration shows the existence of y'. The method then shows only this: on the assumption that y' exists, its value is given by 1). But this assumption of existence makes the demonstration worthless.

3. Many writers of elementary mathematical text-books are not alive to the fact that a demonstration, which involves an assumption of the existence of certain quantities or forms, renders the demonstration invalid. This error of reasoning is extremely common in the calculus. Because *determinate* results are obtained by such reasoning, it is allowed to pass as conclusive.

To show how fallacious this style of reasoning is, let us *assume* that we can write *

$$\frac{1}{x^2-4} = a \sin x + b \cos x.$$

Granting this, it is easy to determine a and b. In fact, setting $x=0$, we get

$$b = -\tfrac{1}{4}.$$

Setting $x = \frac{\pi}{2}$, we get

$$a = \frac{4}{\pi^2 - 16}.$$

Hence

$$\frac{1}{x^2-4} = \frac{4}{\pi^2-16} \sin x - \frac{1}{4} \cos x,$$

a perfectly determinate result; but also a perfectly false result. In fact, the right side of 4) is a periodic function, while the left side is not.

The reader should therefore not begrudge the pains it is sometimes necessary to take, to prove an *existence theorem*. He should also notice that by modifying the form of proof it is sometimes possible to avoid assuming the existence of certain things which enter the demonstration. Witness the demonstrations just given of 1) in 1, 2.

378. *Let* $y=f(x)$, *and* $x=g(t)$. *Let* $g'(t)=\frac{dx}{dt}$ *be finite in* T. *Let* X *be the image of* T. *If* $\frac{dy}{dx}=f'(x)$ *is finite in* X,

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}. \qquad (1$$

---

* In treating the decomposition of a rational function into partial fractions, it is often assumed, *without any justification*, that the decomposition in the form desired is possible.

Before proving this theorem, we wish to illustrate two cases which may occur.

Ex. 1. Let $x = t \sin 2\, m\pi t$. The period of $\sin 2\, m\pi t$ considered as a function of t is $\frac{1}{m}$. By taking m very large but fixed, x will oscillate a great many times near the origin. Where the graph cuts the t-axis, *i.e.* when

$$\Delta t = \pm \frac{1}{2\,m}, \quad \pm \frac{1}{m}, \quad \pm \frac{3}{2\,m}, \cdots$$

we have $\Delta x = 0$.

But however large m is taken, we can determine a $\delta > 0$, such that $\Delta x \neq 0$, in $V_\delta{}^*(0)$. In fact, we have only to take $\delta < \frac{1}{2\,m}$.

What we have shown for $t = 0$ is true for any other point t. That is, we can always choose δ sufficiently small so that in $V_\delta{}^*(t)$, Δx shall not $= 0$.

Ex. 2.
$$\begin{aligned} x &= t^2 \sin \frac{\pi}{t}, \quad \text{for } t \neq 0; \\ &= 0, \qquad\quad \text{for } t = 0. \end{aligned}$$

The graph of this function we considered in 368. For any point $t \neq 0$ we can determine a δ such that in $V_\delta{}^*(t)$, Δx does not vanish. Not so at $t = 0$. Here, however small $\delta > 0$ is taken, x oscillates infinitely often in $V_\delta{}^*(0)$; and thus for an infinity of points in $V_\delta{}^*(0)$, $\Delta x = 0$.

We can, however, throw the points of $V_\delta{}^*(0)$ in two sets. In one, call it V_0, we put the points for which $\Delta x = 0$. Then

$$V_0 = \pm \frac{1}{m}, \quad \pm \frac{1}{m+1}, \quad \cdots$$

In the other set, call it V_1, we put all the other points of V^*. We can now show for the function $y = f(x)$ in the above theorem, that 1) is true for each one of these sets of points, and therefore true for both together.

379. We give now the proof of 378.

Let t be any point in T; let x be the corresponding point in X. Let Δx, Δy be the increments of x, y, corresponding to the increment Δt of t.

Case 1. *There exists a* $V^*(t)$, *in which* $\Delta x \neq 0$.

The identity
$$\frac{\Delta y}{\Delta t} = \frac{\Delta y}{\Delta x} \cdot \frac{\Delta x}{\Delta t} \tag*{(1}$$

does not involve a division by 0, as $\Delta x \neq 0$.

Since $\frac{dx}{dt} = g'(t)$ is finite at t, $\Delta x \doteq 0$ when $\Delta t \doteq 0$. Hence, by 292,

$$\lim_{\Delta t = 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x = 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}.$$

Thus,

$$\lim_{\Delta t = 0} \frac{\Delta y}{\Delta t} = \lim_{\Delta x = 0} \frac{\Delta y}{\Delta x} \cdot \lim_{\Delta t = 0} \frac{\Delta x}{\Delta t},$$

and

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}, \qquad (2$$

which proves the theorem for this case.

Case 2. $\Delta x = 0$ for some point in every $V^*(t)$.

Let V_0 be the points of $V^*(t)$, for which $\Delta x = 0$.

Let V_1 be the remaining points of $V^*(t)$.

If we show

$$\lim \frac{\Delta y}{\Delta t}, \text{ and } \frac{dy}{dx} \lim \frac{\Delta x}{\Delta t}, \qquad (3$$

have one and the same value for every sequence of points whose limit is t, we have proved 2) for this case.

Let A be any sequence in V_0. Then

$$\lim_A \frac{\Delta x}{\Delta t} = 0, \qquad (4$$

since $\Delta x = 0$ for every point in A.

As $\frac{dy}{dx}$ is finite at x,

$$\frac{dy}{dx} \lim_A \frac{\Delta x}{\Delta t} = 0.$$

On the other hand,

$$\lim_A \frac{\Delta y}{\Delta t} = 0.$$

For, Δx being 0 for every point of A, $y = f(x)$ receives no increment, and hence $\Delta y = 0$ in A.

Thus, for every sequence A, the two limits in 3) have the same value, viz. 0.

Let now B be any sequence in V_1 which $\doteq t$. Let the image of the points B be the points C, on the x-axis.

Then, by 292,

$$\lim_B \frac{\Delta y}{\Delta t} = \lim_C \frac{\Delta y}{\Delta x} \cdot \lim_B \frac{\Delta x}{\Delta t}$$

$$= \frac{dy}{dx} \lim_B \frac{\Delta x}{\Delta t}. \tag{5}$$

Thus the two limits of 3) are the same for each sequence B. It remains to show that one is 0. Now, by 4),

$$\lim_B \frac{\Delta x}{\Delta t} = \lim_A \frac{\Delta x}{\Delta t} = 0;$$

since, by hypothesis,

$$\lim \frac{\Delta x}{\Delta t} = g'(t)$$

for any sequence whose limit is the point t.

Hence the right side of 5) is 0.

Thus the two limits 3) have the value 0 for every sequence A or B. These limits therefore have the value 0 for *any* sequence, whether its points all lie in V_0, or in V_1, or partly in V_0 and partly in V_1.

380. The demonstration, as ordinarily given, rests on the identity

$$\frac{\Delta y}{\Delta t} = \frac{\Delta y}{\Delta x} \cdot \frac{\Delta x}{\Delta t}.$$

The theorem is, therefore, only established for functions $x = g(t)$, which fall under Case 1.

If one wishes to give a correct but elementary demonstration, it would suffice to restrict $g(t)$ to have only a finite number of oscillations in an interval T, and have at each point of T a finite differential coefficient. In an elementary text-book on the calculus it is not advisable to consider functions with an infinite number of oscillations.

381. *Let $y = f(x)$ be univariant and continuous. Let $x = g(y)$ be its inverse function. Let $f'(x)$ be finite or infinite in Δ. Let E be the image of Δ. Let x and y be corresponding points in Δ and E.*

If $f'(x)$ *is finite and* $\neq 0$, *then* $g'(y) = \dfrac{1}{f'(x)}$.

If $f'(x) = 0$, *then* $g'(y) = \begin{cases} +\infty \text{ if } f \text{ is increasing.} \\ -\infty \text{ if } f \text{ is decreasing.} \end{cases}$

If $f'(x)$ *is infinite*, $g'(y) = 0$.

Since f is univariant, Δy and therefore also $\dfrac{\Delta y}{\Delta x}$ are $\neq 0$.

Hence the relation

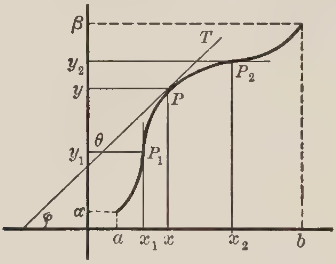
$$\frac{\Delta x}{\Delta y} = \frac{1}{\dfrac{\Delta y}{\Delta x}} \tag{1}$$

does not involve for any point a division by 0.

Since y is continuous, $\Delta y \doteq 0$ when $\Delta x \doteq 0$.

We have therefore only to apply 292 in passing to the limit in 1).

382. The geometric interpretation of 381 is very simple in the following case:



Let $y = f(x)$ be a continuous increasing function in (a, b).

The inverse function $x = g(y)$ is increasing and continuous in (α, β). See Fig.

The graph of $f(x)$ and $g(y)$ is the same curve C. At P_1, P_2 we have points of inflection.

If PT is the tangent at P,

$$\tan \phi = f'(x) = \frac{dy}{dx},$$

$$\tan \theta = g'(y) = \frac{dx}{dy}.$$

Since

$$\theta + \phi = \frac{\pi}{2},$$

$$\tan\theta = \frac{1}{\tan\phi},$$

or

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}.$$

The consideration of the tangents at P_1, P_2 illustrates the theorem for the other cases.

383. We apply the preceding general theorems to find the derivatives of some of the elementary functions, choosing those whose demonstration is often given incorrectly.

$$D_x a^x = a^x \log a. \qquad a > 0,\ x \textit{ arbitrary}. \tag{1}$$

For, let

$$y = a^x.$$

Then

$$\frac{\Delta y}{\Delta x} = a^x \frac{a^{\Delta x} - 1}{\Delta x}. \tag{2}$$

But, by 311,

$$\lim_{\Delta x = 0} \frac{a^{\Delta x} - 1}{\Delta x} = \log a.$$

Passing to the limit in 2), we get 1).

When $a = e$, 1) becomes

$$D_x e^x = e^x. \tag{3}$$

384. 1.

$$D_x \log x = \frac{1}{x}. \qquad x > 0. \tag{1}$$

Let

$$y = \log x.$$

Then

$$x = e^y.$$

But

$$\frac{dx}{dy} = e^y = x. \tag{2}$$

From 2) we get, by 381,

$$\frac{dy}{dx} = \frac{1}{x},$$

which is 1).

2. We can get 1) directly as follows:

$$\frac{\Delta y}{\Delta x} = \frac{\log(x+\Delta x) - \log x}{\Delta x} = \frac{\log\left(1+\frac{\Delta x}{x}\right)}{\Delta x}$$

$$= \frac{1}{x} \cdot \frac{\log\left(1+\frac{\Delta x}{x}\right)}{\frac{\Delta x}{x}}. \tag{3}$$

But, by 310 and 292,

$$\lim_{\Delta x = 0} \frac{\log\left(1+\frac{\Delta x}{x}\right)}{\frac{\Delta x}{x}} = 1. \tag{4}$$

Hence, passing to the limit in 3), we get 1) again.

From 1) we can prove again

$$D_x e^x = e^x. \tag{5}$$

For, from

$$y = e^x,$$

we have

$$x = \log y.$$

Hence, by 1),

$$\frac{dx}{dy} = \frac{1}{y}.$$

Using 381, we have

$$\frac{dy}{dx} = y = e^x,$$

which is 5).

385. 1. *Criticism.* In either of the preceding ways of getting

$$D_x e^x \text{ and } D_x \log x,$$

we need the limit

$$\lim_{u=0} (1+u)^{\frac{1}{u}} = e. \tag{1}$$

Some writers only prove 1) when u runs over the sequence $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots$

Others prove 1) only for a right hand limit. As, however, Δx may have any positive or negative values as it converges to 0, the limit 1) must be established without any restriction.

2. If the method of 384, 2 is used to get $D_x \log x$, we must not only prove 1), but we must show that

$$\lim_{u=0} \log (1+u)^{\frac{1}{u}} = \log \lim_{u=0} (1+u)^{\frac{1}{u}}.$$

This is rarely done.

3. A third method is to employ the Binomial Theorem, which is taken from algebra.

The rigorous demonstration of this theorem for any case, besides that of integral positive exponents, is far beyond the limits of the ordinary high school or college algebra. Moreover, the demonstrations usually given are incorrect. The employment of the Binomial Theorem to find the above derivatives is therefore open to the most serious criticism.

386. 1. The differentiation of the direct circular functions presents nothing of note; let us therefore turn at once to the *inverse* circular functions.

We take

$$y = \text{arc} \sin x \tag{1}$$

as an example. The notation indicates that we have taken the principal branch of arc sin x, [223]. Then

$$-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}. \tag{2}$$

From 1) we have

$$x = \sin y.$$

Hence

$$\frac{dx}{dy} = \cos y = \sqrt{1-x^2}. \tag{3}$$

The radical has the positive sign, as $\cos y$ is not negative for the values 2).

Hence, by 381,

$$\frac{dy}{dx} = D_x \text{arc} \sin x = \frac{1}{\sqrt{1-x^2}}, \qquad |x| \neq 1.$$

$$= +\infty \text{ for } x = \pm 1.$$

2. *Criticism.* In many books the branch of arc sin x which is taken is not specified. Consequently, the sign of the radical in 3) is not specified. For some branches the negative sign should be taken.

387. 1. $D_x x^\mu = \mu x^{\mu-1}$. $x > 0$, μ *arbitrary.* (1

Let
$$y = x^\mu.$$
Then
$$y = e^{\mu \log x}. \tag{2}$$
Let
$$\mu \log x = u.$$
Then
$$y = e^u.$$
But
$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx};$$
and
$$\frac{dy}{du} = e^u, \quad \frac{du}{dx} = \frac{\mu}{x}.$$
Hence
$$\frac{dy}{dx} = \mu x^{\mu-1}.$$

2. *Criticism.* Some writers rest the demonstration on
$$\lim_{u=0} \frac{\log(1+u)}{u};$$
and are thus open to the criticism of 385, 2. Others proceed thus. From 2) we have
$$\log y = \mu \log x.$$

Differentiating both sides, we get
$$\frac{1}{y}\frac{dy}{dx} = \frac{\mu}{x},$$
from which we get 1) at once. This method rests on the assumption that $\frac{dy}{dx}$ exists, and so is open to the criticism of 377, 2.

EXAMPLES

388. 1. $y = a + b\sqrt[3]{x^2} = f(x)$. $b > 0$.

For $x > 0$, we can apply 387, getting, since here $\mu = \frac{2}{3}$,
$$\frac{dy}{dx} = \frac{2}{3}\frac{b}{\sqrt[3]{x}}. \tag{1}$$

2. For $x \leq 0$, the formula of 387 is inapplicable, since it rests on the essential hypothesis that $x > 0$. We can, however, adopt a method applicable to any $x \neq 0$.

Set $x^2 = u$.

Then
$$y = a + bu^{\frac{1}{3}}.$$

For all x in $\mathfrak{R}$ which are $\neq 0$, u is > 0.

Applying 387, we have

$$\frac{dy}{du} = \frac{1}{3} bu^{-\frac{2}{3}}.$$

On the other hand, by 378,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx},$$

since

$$\frac{du}{dx} = 2x$$

is finite.

Hence

$$\frac{dy}{dx} = \frac{2}{3} \frac{b}{\sqrt[3]{x}}.$$

3. When $x = 0$, even this method fails, as u must be > 0, in order to apply 387. In order to calculate the differential coefficient at this point, we must start from its definition.

We have, setting $h = \Delta x$,

$$\frac{\Delta y}{\Delta x} = \frac{f(h) - f(0)}{h} = b \frac{\sqrt[3]{h^2}}{h}.$$

Here, when $\Delta x \doteq 0$,

$$R \lim \frac{\Delta y}{\Delta x} = +\infty, \quad L \lim \frac{\Delta y}{\Delta x} = -\infty.$$

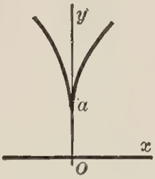


FIG. 1.

The graph of $f(x)$ has thus a vertical cusp at the origin, as in Fig. 1.

4. In order to get
$$Rf'(0),\quad Lf'(0),$$
some readers may be tempted to take the right and left hand limits of the expression 1) for $x=0$. In the present case we would get the correct result. In general, if the expression for $f'(x)$ assumes an indeterminate form for a particular value of x, say $x=a$, the reader must avoid the temptation to conclude that
$$f'(a)=\lim_{x=a} f'(x).$$
This is only true when $f'(x)$ is continuous at a.

Ex. 1.
$$f(x)=x\sin\frac{1}{x},\ x\neq 0;$$
$$=0,\qquad x=0.$$
Here $f'(0)$ does not exist by 367, while, for $x\neq 0$,
$$f'(x)=\sin\frac{1}{x}-\frac{1}{x}\cos\frac{1}{x}.$$
Thus
$$\lim_{x=0} f'(x)$$
also does not exist.

Ex. 2.
$$f(x)=x^2\sin\frac{1}{x},\ x\neq 0;$$
$$=0,\qquad x=0.$$
Here
$$f'(0)=0, \text{ by } 368,$$
while, for $x\neq 0$,
$$f'(x)=2x\sin\frac{1}{x}-\cos\frac{1}{x}.$$
Thus
$$\lim_{x=0} f'(x)$$
does not exist.

5. Let
$$f(x)=x^{\frac{1}{3}}.$$
We find readily that
$$Rf'(0)=Lf'(0)=+\infty.$$
The graph is given in Fig. 2.

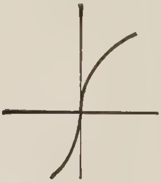


FIG. 2.

389. 1. Let $\log x = l_1 x$, $\log\log x = l_2 x$,

$$\log\log\log x = l_3 x, \text{ etc.}$$

Since $\log u$ is defined only for $u > 0$, we shall suppose that x is taken sufficiently large so that $l_m x$ has a meaning.

We prove now

$$D_x l_m x = \frac{1}{x} \cdot \frac{1}{l_1 x l_2 x \cdots l_{m-1} x}. \qquad m > 1. \tag{1}$$

For, first, let

$$y = l_2 x = \log\log x.$$

Set

$$u = \log x.$$

Then

$$y = \log u.$$

Hence

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{u} \cdot \frac{1}{x} = \frac{1}{x} \frac{1}{\log x} = D_x l_2 x. \tag{2}$$

Next, let

$$y = l_3 x = \log \cdot l_2 x.$$

Set

$$u = l_2 x.$$

Then

$$y = \log u.$$

Hence

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

By 2),

$$\frac{du}{dx} = D_x l_2 x = \frac{1}{x} \frac{1}{\log x}.$$

Hence

$$\frac{dy}{dx} = \frac{1}{u} \cdot \frac{1}{x \log x} = \frac{1}{x l_1 x l_2 x} = D_x l_3 x.$$

By induction, we now establish 1) readily.

2. In a similar manner we establish

$$D_x l_s^{1-\lambda} x = \frac{1 - \lambda}{x l_1 x l_2 x \cdots l_{s-1} x l_s^{\lambda} x}. \qquad \lambda \neq 1. \tag{3}$$

From 1), 3) we have two formulæ to be used later:

$$D_x l_m \frac{1}{x-a} = \frac{-1}{(x-a) l_1 \frac{1}{x-a} \cdots l_{m-1} \frac{1}{x-a}}; \qquad m > 1. \tag{4}$$

and

$$D_x l_m^{1-\lambda} \frac{1}{x-a} = \frac{\lambda - 1}{(x-a) l_1 \frac{1}{x-a} \cdots l_{m-1} \frac{1}{x-a} l_m^{\lambda} \frac{1}{x-a}}. \qquad \lambda \neq 1. \tag{5}$$

In 4), 5) we suppose $x > a$, such that the quantities entering them are defined.

Differentials and Infinitesimals

390. 1. Since

$$f'(x) = \lim \frac{\Delta y}{\Delta x},$$

we have for each $\epsilon > 0$, a $\delta > 0$, such that in $V_\delta^*(x)$,

$$\left| \frac{\Delta y}{\Delta x} - f'(x) \right| < \epsilon;$$

or

$$|\Delta y - f'(x)\Delta x| < \epsilon |\Delta x|,$$

or

$$\Delta y = f'(x)\Delta x + \epsilon' \Delta x, \tag{1}$$

where

$$|\epsilon'| < \epsilon.$$

We call

$$f'(x)\Delta x$$

the differential of $f(x)$, and denote it by

$$dy \quad \text{or} \quad df(x).$$

The relation 1) shows that Δy is made up of two parts, viz.

$$dy \quad \text{and} \quad \epsilon' \Delta x.$$

The ratio of these two parts is

$$\frac{\epsilon'}{f'(x)}. \qquad f'(x) \neq 0.$$

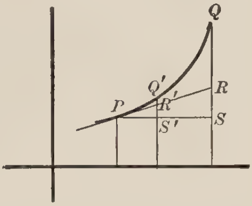
As $f'(x)$ is fixed, for fixed x, and ϵ' can be made numerically as small as we please, by taking δ sufficiently small, we see that the part $\epsilon'\Delta x$ is very small, compared with dy for all points $x+\Delta x$ in V_δ^*. Thus, in the immediate vicinity of x, the *principal part* of Δy is dy.

Differentials owe their importance to this fact.

2. To make the notation homogeneous, it is customary to replace Δx by another symbol, dx, in the expression for dy. We have then

$$dy = f'(x)dx.$$

391. The notion of a differential may be illustrated as follows:



Let the graph of $f(x)$ be that in the figure.

Let PR be the tangent at P; and

$$PS = \Delta x, \quad QS = \Delta y, \quad RS = dy.$$

Then

$$QR = \epsilon'\Delta x.$$

The reader will see, if $dy \neq 0$, that as Q approaches P, QR becomes smaller and smaller as compared with $RS = dy$. This is illustrated by comparing this ratio at Q and at Q'.

We see $dy = RS$ approximates more and more closely to Δy as Q approaches P.

392. A variable whose limit is 0 is called an *infinitesimal.*

When employing differentials, we suppose that the increment given to the independent variable $\Delta x = dx$ can be taken as small, numerically, as we choose. It is thus an infinitesimal. Then both Δy and dy are also infinitesimals.

In the limits considered in 301–304, 310–312, the numerators and denominators furnish examples of infinitesimals.

Also the lengths of the intervals considered in 127, 2, are infinitesimals.

Many other examples of infinitesimals are to be found in the preceding pages, and many more will occur in the following.

The Law of the Mean

393. One of the pillars which support the modern rigorous development of the calculus is the Law of the Mean. It rests on

Rolle's Theorem. Let $f(x)$ be continuous in $\mathfrak{A} = (a, b)$, and $f(a) = f(b)$. Let $f'(x)$ be finite or infinite within $\mathfrak{A}$. Then there exists a point c within $\mathfrak{A}$, for which

$$f'(c) = 0. \qquad a < c < b.$$

Since $f(x)$ is continuous in $\mathfrak{A}$, it is limited, by 350. Its extremes are therefore finite. If f is not constant, one of these extremes is different from the end values.

To fix the ideas, let $\text{Max} f = \mu$ be different from the end values. By 354, there is a point c within $\mathfrak{A}$ such that

$$f(c) = \mu ;$$

while for all points $c + h$ of $\mathfrak{A}$, $h > 0$,

$$f(c+h) - f(c) \leq 0, \quad f(c-h) - f(c) \leq 0.$$

Hence

$$\frac{f(c+h) - f(c)}{h} \leq 0, \tag{1}$$

$$\frac{f(c-h) - f(c)}{-h} \overset{=}{>} 0. \tag{2}$$

According to 1),

$$f'(c) \overset{=}{<} 0 ;$$

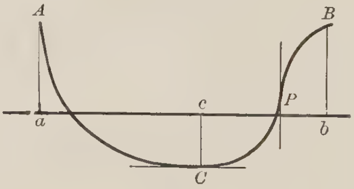
according to 2),

$$f'(c) \overset{=}{>} 0.$$

Those together require that

$$f'(c) = 0.$$

In case $f(x)$ is a constant in $\mathfrak{A}$, the theorem is obviously true.



394. 1. The geometric interpretation of Rolle's theorem is the following:

Let the graph of $f(x)$ be a continuous curve having everywhere a tangent, except possibly at the end points A, B, which are at the same height above or below the x-axis. Then at some point C the tangent is parallel to the x-axis.

Since $f'(x)$ may be infinite, the graph may have points of inflection with vertical tangents, as at P.

2. Let A, B be two points at the same height above the x-axis. The reader will *feel* the truth of Rolle's theorem for simple cases if he tries to draw a continuous curve Γ through A, B, whose tangent is not parallel to the x-axis. Γ should, of course, have no vertical cusp or angular point. We say for *simple* cases, because we cannot draw a curve with an *infinite* number of oscillations or a curve which *does not* have a tangent at A or B. Yet neither of these cases need to be excluded in Rolle's theorem.

395. If $f'(x)$ does not exist for some point within $\mathfrak{A}$, the theorem 394 is not necessarily true, as Fig. 1 shows. (See 366.)

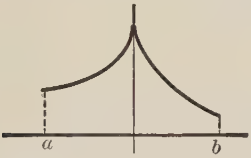


Fig. 1.

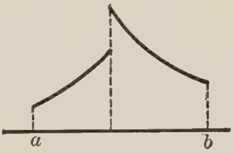


Fig. 2.

If $f'(x)$ is not continuous in $\mathfrak{A}$, the theorem does not need to be true, as Fig. 2 shows.

396. 1. *Criticism.* Many demonstrations are rendered invalid because they rest on the assumption:

1°. In passing from a to b, the function must first increase and then decrease, or first decrease and then increase;

or on the assumption:

2°. There must be at least one point between a, b where the function ceases to increase and begins to decrease, or conversely.

Either of these assumptions is true if we use functions having only a finite number of oscillations in $\mathfrak{A}$.

In case the function has an infinite number of oscillations in $\mathfrak{A}$. neither of the above assumptions need be true.

The function of Ex. 2, 378, where $\mathfrak{A}=(0, 1)$, illustrates the untruth of 1°.

We shall later exhibit functions which oscillate infinitely often in any little interval of $\mathfrak{A}$ and yet have a derivative in $\mathfrak{A}$. Such functions show that 2° is not always correct.

2. The demonstration given in 393 is extremely simple. It rests, however, on the property that a continuous function takes on its extreme values in an interval (a, b). In an elementary treatise this fact might be admitted without proof, since it seems so obvious.

397. 1. *Law of the Mean. Let $f(x)$ be continuous in $\mathfrak{A}=(a, b)$, and let $f'(x)$ be finite or infinite, within $\mathfrak{A}$.*

Then, for some point $a<c<b$,

$$f(b)-f(a)=(b-a)f'(c). \tag{1}$$

Consider the auxiliary function

$$g(x)=f(b)-f(x)-\frac{f(b)-f(a)}{b-a}(b-x).$$

Evidently

$$g(a)=g(b)=0.$$

Also at those points, at which $f'(x)$ is finite,

$$g'(x)=-f'(x)+\frac{f(b)-f(a)}{b-a}; \tag{2}$$

while at the other points of $\mathfrak{A}$, $g'(x)$ is infinite. Thus $g(x)$ is continuous in $\mathfrak{A}$, and $g'(x)$ is finite or infinite within $\mathfrak{A}$. Hence, for some point $a<c<b$,

$$g'(c)=0, \quad \text{by 393.} \tag{3}$$

Setting $x=c$ in 2), and using 3), we get

$$f'(c)=\frac{f(b)-f(a)}{b-a}, \tag{4}$$

which is 1).

2. The relation 1) is commonly written as follows:

Set
$$b - a = h;$$

then, since c lies *within* (a, b),

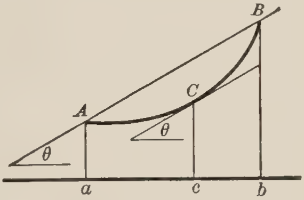
$$c = a + \theta h. \qquad 0 < \theta < 1.$$

Thus 1) gives

$$f(a+h) = f(a) + hf'(a+\theta h).$$

3. The reader should observe that although $f'(x)$ may be infinite within $\mathfrak{A}$, the point c in 1) is such that $f'(c)$ is finite.

398. The following is the geometric interpretation of the Law of the Mean. Let ACB be the graph of $f(x)$ in $\mathfrak{A}$. Let the chord AB make the angle θ with the x-axis. Then



$$\tan\theta = \frac{f(b) - f(a)}{b - a}.$$

Also by 397, 4),

$$f'(c) = \tan\theta.$$

That is: at some point c within the interval (a, b) the tangent is parallel to the chord AB.

399. When either of the conditions that enter the Law of the Mean are violated, the point c may not exist. This is illustrated by the following.

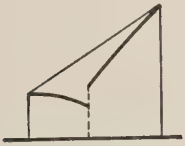


FIG. 1.

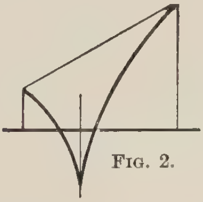


FIG. 2.

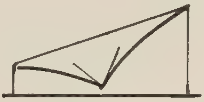


FIG. 3.

1. $f(x)$ is not continuous in (a, b). Fig. 1.

2. The differential coefficient does not exist at some point within (a, b). Figs. 2, 3.

400. We give now some elementary applications of the Law of the Mean.

1. *Let* $f(x)$ *be continuous in* $\mathfrak{A} = (a, b)$; *and let its derivative* $f'(x) = \lambda$, *a constant, within* $\mathfrak{A}$. *Then*

$$f(x) = \lambda x + \mu, \quad \text{in } \mathfrak{A}. \qquad (1$$

Let $x > a$ be a point of $\mathfrak{A}$. The function $f(x)$ satisfies the conditions of the Law of the Mean in $\mathfrak{B} = (a, x)$.

Hence, by 397,

$$f(x) = f(a) + (x - a)f'(c). \qquad a < c < x. \qquad (2$$

Since by hypothesis,

$$f'(c) = \lambda,$$

we get 1) from 2), on setting

$$\mu = f(a) - af'(c).$$

Since, by hypothesis, $f(x)$ is continuous, the formula 1) is also true for $x = a$.

2. As a corollary of 1 we have:

Let $f(x)$ *be continuous in* $\mathfrak{A} = (a, b)$ *and let its derivative be* 0 *within* $\mathfrak{A}$. *Then* $f(x)$ *is a constant in* $\mathfrak{A}$.

3. *Let* $f(x)$, $g(x)$ *be continuous in the interval* $\mathfrak{A}$, *and let* $f'(x) = g'(x)$ *within* $\mathfrak{A}$. *Then, C being some constant,*

$$f(x) = g(x) + C, \quad \text{in } \mathfrak{A}. \qquad (3$$

For

$$k(x) = f(x) - g(x)$$

satisfies the conditions of 2. Hence

$$k(x) = C, \quad \text{in } \mathfrak{A}.$$

401. *Let* $f(x)$ *be continuous in* $\mathfrak{A} = (a, b)$, *while* $f'(x)$ *is finite or infinite within* $\mathfrak{A}$. *Let* $f'(x)$, *when not* 0, *have one sign* σ. *Then* f *is monotone increasing in* $\mathfrak{A}$, *if* σ *is positive; monotone decreasing, if* σ *is negative.*

Let

$$a \leq x' < x'' \leq b.$$

By the Law of the Mean,

$$f(x'') = f(x') + (x'' - x')f'(c). \qquad x' < c < x''.$$

As $x'' - x' > 0$, and $f'(c)$ has the sign σ, when not 0,

$$f(x'') \geqq f(x'), \text{ if } \sigma \text{ is positive ;}$$

$$f(x'') \leqq f(x'), \text{ if } \sigma \text{ is negative.}$$

402. *Criticism.* Some writers state that $f(x)$ is increasing when $f'(x)$ is positive, and conversely when $f(x)$ is increasing $f'(x)$ is positive.

The second statement is not true, as the figure shows. P is a point of inflection, with a tangent parallel to the x-axis.

The error in the reasoning is instructive. By definition, if $f(x)$ is increasing,

$$\frac{\Delta f}{\Delta x}$$

is positive. It is now inferred that therefore

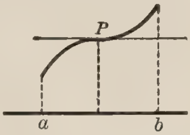
$$\lim \frac{\Delta f}{\Delta x} = f'(x)$$

is positive. All one can strictly infer is that $f'(x) \geqq 0$.

The function

$$y = x^3, \qquad \mathfrak{A} = (-1, 1)$$

illustrates this, at the point $x = 0$. See Fig.



403. *Let $f(x)$ be continuous in $\mathfrak{A} = (a, b)$, and $f'(x)$ finite or infinite within $\mathfrak{A}$. $f'(x)$ shall not vanish for all the points of any subinterval $\mathfrak{B} = (\alpha, \beta)$ of $\mathfrak{A}$. When not 0, let $f'(x)$ have always one sign σ in $\mathfrak{A}$. Then $f(x)$ is an increasing or a decreasing function in $\mathfrak{A}$, according as σ is positive or negative.*

To fix the ideas, let σ be positive.

Let $$a \leq x' < x'' \leq b.$$

By the Law of the Mean,

$$f(x'') = f(x') + (x'' - x')f'(c). \qquad x' < c < x''.$$

As $x'' - x' > 0$, and $f'(c) \gtreqless 0$, we have

$$f(x'') \geq f(x');$$

i.e., $f(x)$ is *monotone* increasing in $\mathfrak{A}$. To show it is *constantly* increasing in $\mathfrak{A}$, suppose

$$f(\alpha) = f(\beta).$$

Then $f(x)$ must $= f(\alpha)$ for *all* points in $\mathfrak{B} = (\alpha, \beta)$, since it is a monotone increasing function.

Since $f(x)$ is a constant in $\mathfrak{B}$, $f'(x) = 0$ in $\mathfrak{B}$, which contradicts the hypothesis.

404. *Let* $f'(x)$ *be continuous in the interval* $\mathfrak{A}$. *Then the difference quotient* $\frac{\Delta y}{\Delta x}$ *converges uniformly to* $f'(x)$ *in* $\mathfrak{A}$.

For, by the Law of the Mean,

$$f(x+h) - f(x) = hf'(x + \theta h). \qquad 0 < \theta < 1.$$

Hence

$$\frac{\Delta y}{\Delta x} = f'(x + \theta h). \tag{1}$$

But $f'(x)$ being continuous in $\mathfrak{A}$, is uniformly continuous by 352. Hence

$$\lim_{h=0} f'(x + \theta h) = f'(x). \quad \textit{uniformly}.$$

Hence, by 1),

$$\lim \frac{\Delta y}{\Delta x} = f'(x). \quad \textit{uniformly in } \mathfrak{A}.$$

Derivatives of Higher Order

405. The first derivative of $f'(x)$ is called the *second derivative of* $f(x)$, and is denoted by

$$f''(x),\ D_x^2 f(x),\ \frac{d^2 f}{dx^2}.$$

Evidently,

$$f''(x) = \lim_{h=0} \frac{f'(x+h) - f'(x)}{h} = \lim_{\Delta x = 0} \frac{\Delta f'(x)}{\Delta x};$$

this limit being finite or infinite.

In this way we may continue to form third, fourth, ⋯ and derivatives of any order.

Derivatives of order n are denoted by

$$f^{(n)}(x),\quad D_x^n f(x),\quad \frac{d^n f}{dx^n}.$$

406. We add the following formulæ, which will be used later. They are easily verified:

$$D_x^n x^\mu = \mu\cdot\mu-1\cdot\cdots\mu-n+1\cdot x^{\mu-n}. \qquad x>0. \qquad (1$$

$$D_x^n(1+x)^\mu = \mu\cdot\mu-1\cdot\cdots\mu-n+1\cdot(1+x)^{\mu-n}. \quad 1+x>0. \qquad (2$$

$$D_x^n e^x = e^x. \qquad (3$$

$$D_x^n \sin x = \sin\left(\frac{n\pi}{2}+x\right). \qquad (4$$

$$D_x^n \cos x = \cos\left(\frac{n\pi}{2}+x\right). \qquad (5$$

407. Let $y=uv$, where u, v have derivatives of any desired order. The following relation is known as *Leibnitz's formula*.

$$y^{(n)} = u^{(n)}v + \binom{n}{1}u^{(n-1)}v' + \binom{n}{2}u^{(n-2)}v'' + \cdots$$

$$+\binom{n}{2}u''v^{(n-2)} + \binom{n}{1}u'v^{(n-1)} + uv^{(n)}, \qquad (1$$

where

$$\binom{n}{m} = \frac{n\cdot n-1\cdot\cdots n-m+1}{1\cdot 2\cdots m}.$$

We prove it by complete induction; *i.e.* we assume it true for n and prove it is true for $n+1$. For $n=1, 2$ it is obviously true.

Differentiating 1), we get

$$y^{(n+1)} = u^{(n+1)}v + \binom{n}{1}u^{n}v' + \binom{n}{2}u^{(n-1)}v'' + \binom{n}{3}u^{(n-2)}v''' + \cdots$$

$$+ u^{(n)}v' + \binom{n}{1}u^{(n-1)}v'' + \binom{n}{2}u^{(n-2)}v''' + \cdots \qquad (2$$

Now, by 96,

$$\binom{n}{m}+\binom{n}{m-1}=\binom{n+1}{m}.$$

This in 2), gives

$$y^{(n+1)}=u^{(n+1)}v+\binom{n+1}{1}u^{(n)}v'+\binom{n+1}{2}u^{(n-1)}v''+\cdots$$

which is 1), when we replace in it n by $n+1$.

408. 1. Let us apply Leibnitz's formula to find the derivatives of

$$y=f(x)=\tan x.$$

We have

$$y'=\sec^2 x,$$

$$y''=2\sec^2 x\tan x=2\,yy'.$$

Now

$$D_x^n(yy')=y^{(n+1)}y+\binom{n}{1}y^{(n)}y'+\binom{n}{2}y^{(n-1)}y''+\cdots$$

This gives

$$y'''=2\,(y''y+y'^2)\,;$$

$$y^{\text{iv}}=2\,(y'''y+3\,y'y'')\,;$$

$$y^{\text{v}}=2(y^{\text{iv}}y+4\,y'''y'+3\,y''^2),\ \text{etc.}$$

2. Another way is the following, which will lead us to a formula that we shall need later.

We have

$$y\cos x=\sin x\,;$$

or setting,

$$u=\sin x,\ \ z=\cos x,$$

$$u=yz.$$

Now by Leibnitz's formula,

$$u^{(n)}=y^{(n)}z+\binom{n}{1}y^{(n-1)}z'+\binom{n}{2}y^{(n-2)}z''+\cdots \tag{1}$$

Also, by 406, 4), 5),

$$u^{(n)} = \sin\left(\frac{n\pi}{2} + x\right), \quad z^{(n)} = \cos\left(\frac{n\pi}{2} + x\right).$$

Hence 1) gives

$$\sin\left(\frac{n\pi}{2} + x\right) = \left\{ y^{(n)} - \binom{n}{2} y^{(n-2)} + \binom{n}{4} y^{(n-4)} - \cdots \right\} \cos x$$
$$- \left\{ \binom{n}{1} y^{(n-1)} - \binom{n}{3} y^{(n-3)} + \binom{n}{5} y^{(n-5)} - \cdots \right\} \sin x.$$

This gives the recursion formula,

$$y^{(n)} = \frac{\sin\left(\frac{n\pi}{2} + x\right)}{\cos x} + \binom{n}{2} y^{(n-2)} - \binom{n}{4} y^{(n-4)} + \cdots \tag{2}$$
$$+ \tan x \left\{ \binom{n}{1} y^{(n-1)} - \binom{n}{3} y^{(n-3)} + \cdots \right\}.$$

Setting $x = 0$ in 2), we get

$$f^{(n)}(0) - \binom{n}{2} f^{(n-2)}(0) + \binom{n}{4} f^{(n-4)}(0) - \cdots = \sin\frac{n\pi}{2}. \tag{3}$$

Taylor's Development in Finite Form

409. 1. Using derivatives of higher order, we can generalize the Law of the Mean as follows:

In the interval $\mathfrak{A} = (a, b)$, *let* $f(x)$ *and its first* $n - 1$ *derivatives be continuous. Let* $f^{(n)}(x)$ *be finite or infinite within* $\mathfrak{A}$. Then for any x in $\mathfrak{A}$,

$$f(x) = f(a) + \frac{(x-a)}{1!} f'(a) + \cdots$$
$$+ \frac{(x-a)^{n-1}}{n-1!} f^{(n-1)}(a) + \frac{(x-a)^n}{n!} f^{(n)}(c), \tag{1}$$

where

$$a < c < x.$$

As in 397, we introduce an auxiliary function

$$g(u)=f(x)-f(u)-(x-u)f'(u)-\frac{(x-u)^2}{2\,!}f''(u)-\cdots$$

$$-\frac{(x-u)^{n-1}}{n-1\,!}f^{(n-1)}(u)-\frac{(x-u)^n}{n\,!}A, \tag{2}$$

where A is independent of u. This function is obviously a continuous function of u in $\mathfrak{A}$ for any x in $\mathfrak{A}$. Differentiating with respect to u, we get, observing terms cancel in pairs,

$$g'(u)=\frac{(x-u)^{n-1}}{n-1\,!}\{-f^{(n)}(u)+A\}, \tag{3}$$

for any u *within* $\mathfrak{A}$.

Thus the derivative of $g(u)$ is finite within $\mathfrak{A}$.

To apply Rolle's theorem to $g(u)$, with reference to the interval $\mathfrak{B}=(a,x)$, it is only necessary to determine A in 2) so that

$$g(a)=g(x).$$

But obviously

$$g(x)=0.$$

We therefore suppose A so chosen that

$$0=f(x)-f(a)-(x-a)f'(a)-\cdots$$

$$-\frac{(x-a)^{n-1}}{n-1\,!}f^{(n-1)}(a)-\frac{(x-a)^n}{n\,!}A. \tag{4}$$

Then by Rolle's theorem, there is a point $a<c<x$, such that

$$g'(c)=0.$$

This in 3) gives

$$\frac{(x-c)^{n-1}}{n-1\,!}\{f^{(n)}(c)-A\}=0. \tag{5}$$

As $c\neq x$, the first factor in 5) is not 0. Hence the parenthesis is 0, which gives

$$f^{(n)}(c)=A.$$

Putting this value of A in 4), we have 1).

2. The formula 1) is called *Taylor's development of $f(x)$ in finite form.*

It may also be written as follows:

Set

$$x = a + h, \quad c = a + \theta h. \qquad 0 < \theta < 1.$$

Then 1) becomes

$$f(a+h) = f(a) + \frac{h}{1!} f'(a) + \frac{h^2}{2!} f''(a) + \cdots$$

$$+ \frac{h^{n-1}}{n-1!} f^{(n-1)}(a) + \frac{h^n}{n!} f^{(n)}(a + \theta h). \tag{6}$$

$$a + h, \text{ in } \mathfrak{A}.$$

410. 1. *Let $f(x)$ and its first $n-1$ derivatives be continuous in the interval $\mathfrak{B} = (a - H, a + H)$, while $f^{(n)}(x)$ is finite in $\mathfrak{B}$. Then for any x in $\mathfrak{B}$,*

$$f(x) = f(a) + \frac{(x-a)}{1!} f'(a) + \frac{(x-a)^2}{2!} f''(a) + \cdots$$

$$+ \frac{(x-a)^{n-1}}{n-1!} f^{(n-1)}(a) + \frac{(x-a)^n}{n!} f^{(n)}(c), \tag{1}$$

$$= f(a+h) = f(a) + \frac{h}{1!} f'(a) + \frac{h^2}{2!} f''(a) + \cdots$$

$$+ \frac{h^{n-1}}{n-1!} f^{(n-1)}(a) + \frac{h^n}{n!} f^{(n)}(a + \theta h), \tag{2}$$

where

$$x = a + h, \quad c = a + \theta h, \quad 0 < \theta < 1, \quad |h| \leq H.$$

The truth of this theorem for the left hand half of $\mathfrak{B}$ follows from the fact that the reasoning of 409 does not depend upon a being $< b$; it holds when $a > b$, if we change x and c accordingly.

2. When $a = 0$, 1) gives

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \cdots + \frac{x^n}{n!} f^{(n)}(\theta x). \tag{3}$$

$$0 < \theta < 1.$$

This is known as *Maclaurin's development.*

411. 1. Let $f(x)=\sin x$.

From 410, 3) we get

$$\sin x = x - x^2 \frac{\sin \theta x}{2\,!} \qquad (1$$

$$= x - \frac{x^3}{3\,!} + \frac{x^4}{4\,!}\sin \theta x, \qquad 0<\theta<1.$$

$$= x - \frac{x^3}{3\,!} + \frac{x^5}{5\,!} - \frac{x^6}{6\,!}\sin \theta x,$$

etc.

The θ's in these formulæ are not necessarily the same. Let

$$0<x<\frac{\pi}{2}.$$

These formulæ show then that

$$\sin x < x$$

$$> x - \frac{x^3}{3\,!}$$

$$< x - \frac{x^3}{3\,!} + \frac{x^5}{5\,!},$$

etc.

From 1) we have again

$$\lim_{x=0} \frac{\sin x}{x} = 1 - \lim_{x=0} x \cdot \frac{\sin \theta x}{2} = 1.$$

See 301.

412. Let

$$f(x) = \cos x.$$

As before, we get

$$\cos x < 1$$

$$> 1 - \frac{x^2}{2\,!}, \qquad 0<x<\frac{\pi}{2}.$$

$$< 1 - \frac{x^2}{2\,!} + \frac{x^4}{4\,!},$$

etc.

From

$$\cos x = 1 - \frac{x^2}{2\,!} + \frac{x^3}{3\,!}\sin \theta x,$$

we have

$$\lim_{x=0} \frac{1-\cos x}{x^2} = \frac{1}{2} - \frac{1}{3\,!} \lim_{x=0} x \sin \theta x$$

$$= \tfrac{1}{2}.$$

See 304.

413. 1. Let

$$f(x) = \log (1+x).$$

From 410, 3) we get

$$\log (1+x) = x - \frac{x^2}{2\,!} \frac{1}{(1+\theta x)^2}. \qquad 0<\theta<1.$$

Hence, as in 310,

$$\lim_{x=0} \frac{\log (1+x)}{x} = 1.$$

2. Let

$$f(x) = a^x. \qquad a>0.$$

From 410, 3) we have

$$a^x = 1 + \frac{x}{1\,!} \log a + \frac{x^2}{2\,!} \log^2 a \cdot a^{\theta x}. \qquad 0<\theta<1.$$

Hence, as in 311,

$$\lim_{x=0} \frac{a^x - 1}{x} = \log a.$$

3. Let

$$f(x) = (1+x)^\mu.$$

We get from 410, 3)

$$(1+x)^\mu = 1 + \mu x + x^2 \cdot \frac{\mu \cdot \mu - 1}{2} (1+\theta x)^{\mu-2}. \qquad |x|<1, \ 0<\theta<1.$$

Hence, as in 312,

$$\lim_{x=0} \frac{(1+x)^\mu - 1}{x} = \mu.$$

FUNCTIONS OF SEVERAL VARIABLES

Partial Differentiation

414. The definition of a partial differential coefficient and a partial derivative of a function of several variables is analogous to the corresponding definitions for functions of a single variable.

Let $f(x_1 \cdots x_m)$ be defined over a domain D, for which $x = (x_1 \cdots x_m)$ is a proper limiting point.

Let $x' = (x_1 \cdots x_{\iota-1}, x_\iota + h, x_{\iota+1} \cdots x_m)$ be any point of D, different from x. If

$$\eta = \lim_{h=0} \frac{f(x') - f(x)}{h} \tag{1}$$

is finite or infinite, it is called the (first) *partial differential coefficient of f with respect to x_ι at the point x.* The aggregate of these η's defines a new function over a certain domain $\Delta \leqq D$, which is called the (first) *partial derivative of f with respect to x_ι.*

It is denoted variously by

$$D_{x_\iota} f(x_1 \cdots x_m), \quad \frac{\partial f(x_1 \cdots x_m)}{\partial x_\iota}, \quad f'_{x_\iota}(x_1 \cdots x_m). \tag{2}$$

When h in 1) is restricted to positive values, η is called a *right hand partial differential coefficient*, and their aggregate gives rise to the *right hand partial derivative with respect to x_ι.*

The meaning of the terms *left hand partial differential coefficient* and *derivative with respect to x_ι* is obvious.

They are denoted by putting the letters R and L before the symbols 2).

The function $f(x_1 \cdots x_m)$ has therefore in general m (first) partial derivatives,

$$\frac{\partial f}{\partial x_1}, \quad \frac{\partial f}{\partial x_2}, \quad \cdots \quad \frac{\partial f}{\partial x_m}.$$

The process of obtaining these partial derivatives is called *partial differentiation.*

415. Example.

$$f(xy) = \sqrt{x^2 + y^2}.$$

If the point x, y is not at the origin,

$$x^2 + y^2 > 0.$$

We can therefore apply 378 and 387, getting

$$\frac{\partial f}{\partial x} = \frac{x}{\sqrt{x^2 + y^2}}, \quad \frac{\partial f}{\partial y} = \frac{y}{\sqrt{x^2 + y^2}}.$$

Thus the partial derivatives with respect to x and y exist at all points different from the origin.

When the point x, y is at the origin, we cannot apply this method. (Compare 388.)

We therefore proceed directly. We have

$$\frac{\Delta f}{\Delta x} = \frac{\sqrt{\Delta x^2}}{\Delta x}.$$

This shows that

$$Rf'_x(0, 0) = +1, \quad Lf'_x(0, 0) = -1.$$

Thus the partial differential coefficient with respect to x does not exist at the origin. Similarly,

$$Rf'_y(0, 0) = +1, \quad Lf'_y(0, 0) = -1;$$

and the partial derivative with respect to y does not exist at the origin.

416. In the case of two independent variables, the (first) partial differential coefficients admit a simple geometric interpretation.

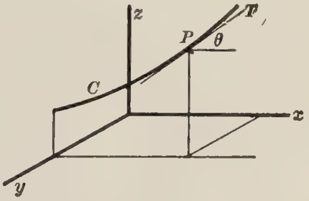
Let the graph of

$$z = f(x, y)$$

be a surface S. The plane

$$y = \text{constant}$$

intersects S in a curve C.



Let PT be the tangent to C at $P = (x, y, z)$, making the angle θ with the x-axis. Then

$$\frac{\partial f}{\partial x} = \tan \theta.$$

Compare 365.

The partial differential coefficient

$$\frac{\partial f}{\partial y}$$

has a similar meaning with respect to the y-axis.

417. Let $f'_{x_\iota}(x_1 \cdots x_m)$ be finite for a domain Δ. We may now reason on f'_{x_ι} as we did on f. Let x be a proper limiting point of Δ, and $x' = (x_1 \cdots x_{\jmath-1}, x_\jmath + h, x_{\jmath+1} \cdots x_m)$ any point of Δ different from x.

If

$$\lim_{h=0} \frac{f'_{x_\iota}(x') - f'_{x_\iota}(x)}{h} = \eta$$

is finite or infinite, η is called the *second partial differential coefficient* of f with respect to x_ι, x_j at the point x, and is denoted by

$$D^2{}_{x_\iota x_j} f(x_1 \cdots x_m), \quad \frac{\partial^2 f(x_1 \cdots x_m)}{\partial x_j \partial x_\iota}, \quad f'_{x_\iota x_j}(x_1 \cdots x_m).$$

The aggregate of these η's will define a new function over a certain domain $\Delta_1 \overline{\gtrless} \Delta$, which is called the *second partial derivative, first with respect to* x_ι, *then with respect to* x_j.

Proceeding in this way, we may form *third, fourth,* $\cdots$ *partial differential coefficients and derivatives.*

Change in the Order of Differentiating

418. 1. In almost all cases which occur in practice, the partial differential coefficient has the same value, however the order of differentiation is chosen. For example:

$$f'''_{x_1x_2x_3} = f'''_{x_1x_3x_2} = f'''_{x_2x_1x_3} = f'''_{x_2x_3x_1} = f'''_{x_3x_1x_2} = f'''_{x_3x_2x_1}.$$

That this is not always true is shown by the following example:

2. $$f(x, y) = xy\frac{x^2 - y^2}{x^2 + y^2}, \quad \textit{for points different from the origin.}$$

$$= 0, \quad \textit{for the origin.}$$

Then if x, y is not the origin,

$$\frac{\partial f}{\partial x} = y\left\{\frac{x^2 - y^2}{x^2 + y^2} + \frac{4\,x^2y^2}{(x^2 + y^2)^2}\right\}, \tag{1}$$

$$\frac{\partial f}{\partial y} = x\left\{\frac{x^2 - y^2}{x^2 + y^2} - \frac{4\,x^2y^2}{(x^2 + y^2)^2}\right\}. \tag{2}$$

At the origin,

$$\frac{\partial f}{\partial x} = 0, \quad \frac{\partial f}{\partial y} = 0. \tag{3}$$

From 1), 2) we have, in particular,

$$f'_x(0, y) = -y, \quad y \neq 0; \tag{4}$$

$$f'_y(x, 0) = x, \qquad x \neq 0. \tag{5}$$

Consider now the second partial derivatives.

From 1), 2) we have, for all points different from the origin,

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{x^2 - y^2}{x^2 + y^2}\left\{1 + \frac{8x^2y^2}{(x^2+y^2)^2}\right\} = \frac{\partial^2 f}{\partial y \partial x}.$$

At the origin, we have from 3), 4), 5),

$$\frac{\Delta f'_x}{\Delta y} = -\frac{\Delta y}{\Delta y}; \quad \therefore f''_{xy}(0, 0) = -1. \tag{6}$$

$$\frac{\Delta f'_y}{\Delta x} = \frac{\Delta x}{\Delta x}; \quad \therefore f''_{yx}(0, 0) = +1. \tag{7}$$

Hence, at the origin,

$$\frac{\partial^2 f}{\partial x \partial y} \neq \frac{\partial^2 f}{\partial y \partial x}.$$

3. In connection with this example, we may warn the inexperienced reader to avoid certain errors he is likely to fall into.

To get the equations 3), *i.e.*

$$f'_x(0, 0) = 0, \quad f'_y(0, 0) = 0,$$

it is *not permissible* to set $x = 0$, $y = 0$ in the relations 1), 2). In fact, these formulæ were obtained under the express stipulation that this point $x = y = 0$ be ruled out.

To get the equation 6), *i.e.*

$$f''_{xy}(0, 0) = -1, \tag{6}$$

it is not permissible to differentiate 4), *i.e.*

$$f'_x(0, y) = -y, \tag{4}$$

with respect to y, thus getting

$$f''_{xy}(0, y) = -1;$$

and in this set $y = 0$, getting the required value of $f''_{xy}(0, 0)$.

In fact, the relation 4) was obtained under the express condition that $y \neq 0$.

Similar remarks apply to $f''_{yx}(0, 0)$.

Junior students are so accustomed to differentiate with their eyes shut that they often overlook the fact that formulæ and theorems are usually not universally true, but are subject to more or less stringent conditions. Compare also the example of 388, 4.

419. It is easy to see *a priori* why $f''_{xy}(a, b)$ *may be* different from $f''_{yx}(ab)$.

By definition,

$$f'_x(a, y) = \lim_{h=0} \frac{f(a+h, y) - f(a, y)}{h},$$

$$f''_{xy}(a, b) = \lim_{k=0} \frac{f'_x(a, b+k) - f'_x(a, b)}{k}$$

$$= \lim_{k=0} \frac{1}{k} \left\{ \lim_{h=0} \frac{f(a+h, b+k) - f(a, b+k)}{h} \right.$$

$$\left. - \lim_{h=0} \frac{f(a+h, b) - f(a, b)}{h} \right\}. \qquad (1$$

Let us set

$$F(h, k) = \frac{f(a+h, b+k) - f(a, b+k) - f(a+h, b) + f(a, b)}{hk}.$$

Then 1) gives

$$f''_{xy}(a, b) = \lim_{k=0} \lim_{h=0} F(h, k).$$

In a similar manner we find that

$$f''_{yx}(a, b) = \lim_{h=0} \lim_{k=0} F(h, k).$$

These formulæ show that $f''_{xy}(a, b)$, $f''_{yx}(a, b)$ are double iterated limits, taken in different order. It is therefore not astonishing that a change in the order of passing to the limit may produce a change in the result. Cf. 322, 323.

420. 1. We consider now certain cases when it is possible to change the order of differentiation in a partial differential coefficient.

Let $f(xy)$ be defined in $D(a, b)$. Let D^* be the deleted domain of D. We suppose:

α) that f'_x exists in D,

β) that f''_{xy} exists in D^*,

γ) that $\lim\limits_{x=a,\, y=b} f''_{xy} = \lambda$. finite or infinite.

Then

$$f''_{xy}(a, b) = \lambda. \tag{1}$$

If, moreover,

δ) f'_y exists for all points of D on the line $y = b$; then

$$f''_{yx}(a, b) = \lambda. \tag{2}$$

We suppose first, that all four conditions α–δ are satisfied, and show that then

$$f''_{xy}(a, b) = f''_{yx}(a, b). \tag{2'}$$

Let

$$F = \frac{f(a+h, b+k) - f(a, b+k) - f(a+h, b) + f(ab)}{hk},$$

as in 419. We introduce the auxiliary functions

$$G(x) = f(x, b+k) - f(x, b), \tag{3}$$

$$H(y) = f(a+h, y) - f(a, y). \tag{4}$$

Then

$$hkF = G(a+h) - G(a) \tag{5}$$

$$= H(b+k) - H(b). \tag{6}$$

Setting, as usual,

$$h = \Delta x, \quad k = \Delta y,$$

we have from δ),

$$\left|\frac{\Delta f}{\Delta y} - f'_y(x, b)\right| = \epsilon(x);$$

where $\epsilon = \epsilon(x)$ is a function of k and x, such that

$$\lim_{k=0} \epsilon = 0. \tag{7}$$

Hence

$$G(x) = \Delta f = k\{f_y'(x, b) + \epsilon(x)\}. \qquad (8$$

Similarly, by α),

$$H(y) = h\{f_x'(a, y) + \eta(y)\}; \qquad (9$$

where $\eta = \eta(y)$ is a function of h and y, such that

$$\lim_{h=0} \eta = 0. \qquad (10$$

Then 5), 8) give

$$F = \frac{1}{h}\{f_y'(a+h, b) - f_y'(a, b) + \epsilon(a+h) - \epsilon(a)\}. \qquad (11$$

Similarly, 6), 9) give

$$F = \frac{1}{k}\{f_x'(a, b+k) - f_x'(a, b) + \eta(b+k) - \eta(b)\}. \qquad (12$$

On the other hand, we can apply the Law of the Mean to 5), by virtue of α), getting

$$kF = G'(c). \qquad a < c < a+h, \text{ or } a+h < c < a. \qquad (13$$

Differentiating 3), we get, using 13),

$$kF = \{f_x'(c, b+k) - f_x'(c, b)\}. \qquad (14$$

By virtue of β, we can apply the Law of the Mean to 14), getting

$$F = f_{xy}''(c, d). \qquad b < d < b+k, \text{ or } b+k < d < b. \qquad (15$$

From 11), 15) we have

$$f_{xy}''(c, d) = \frac{f_y'(a+h, b) - f_y'(a, b)}{h} + \frac{\epsilon(a+h)}{h} - \frac{\epsilon(a)}{h}. \qquad (16$$

From 12), 15) we have

$$f_{xy}''(c, d) = \frac{f_x'(a, b+k) - f_x'(ab)}{k} + \frac{\eta(b+k)}{k} - \frac{\eta(b)}{k}. \qquad (17$$

We can now apply 324 to 16).

Now, by γ),

$$\lim_{h=0, k=0} f_{xy}''(c, d) = \lambda.$$

By 7)
$$\lim_{k=0} \epsilon(a+h) = 0, \quad \lim_{k=0} \epsilon(a) = 0.$$

Hence, letting k first pass to the limit, and then h,

$$\lambda = \lim_{h=0} \lim_{k=0} \left[\frac{f'_y(a+h, b) - f'_y(ab)}{h} + \frac{\epsilon(a+h)}{h} - \frac{\epsilon(a)}{h} \right]$$
$$= \lim_{h=0} \frac{f'_y(a+h, b) - f'_y(ab)}{h}$$
$$= f''_{yx}(a, b). \tag{18}$$

Similarly, 17) gives, letting first h pass to the limit, and then k,

$$\lambda = f''_{xy}(a, b). \tag{19}$$

The equations 18), 19) prove 2′).

2. If we wish to prove 1), without imposing the condition δ), we have only to observe that 17) has been established without reference to δ). But, as has just been shown, we can conclude 1) from 17).

3. It is well to note that this demonstration does not postulate the *existence* of f''_{yx}; or the *continuity* of either of the second partial derivatives; or the *continuity* of f'_y in D or D^*.

We observe also that x and y can obviously be interchanged in the statement of the above theorem.

421. The case which ordinarily arises is embodied in the following *corollary:*

Let f'_x, f'_y, f''_{xy} be continuous in the domain of the point a, b. Then $f''_{yx}(a, b)$ exists, and is equal to $f''_{xy}(a, b)$.

422. It is easy to generalize 421 as follows:

Let the partial derivatives of $f(x_1 \cdots x_m)$ of order $\overline{\overline{<}}\, n$ be continuous in the domain of the point x. Then we can permute the indices ι in

$$f^{(n)}_{x_{\iota_1} x_{\iota_2} \cdots x_{\iota_n}}(x_1 \cdots x_m), \tag{1}$$

without changing its value.

Since any permutation of the n indices

$$\iota_1'\iota_2'\cdots\iota_n'$$

can be obtained from any other permutation

$$\iota_1\iota_2\cdots\iota_n,$$

by repeated interchanges of successive indices, we have only to show that we can interchange any two successive indices as ι_r, ι_{r+1} in 1) without changing its value.

Let us introduce the function of x_{ι_r}, $x_{\iota_{r+1}}$

$$g(x_{\iota_r}, x_{\iota_{r+1}}) = f^{(r-1)}_{\iota_1\cdots\iota_{r-1}}(x_1\cdots x_m),$$

where we consider all the variables on the right as fixed, except the two noted in g.

Then, by 421,

$$\frac{\partial^2 g}{\partial x_{\iota_r}\partial x_{\iota_{r+1}}} = \frac{\partial^2 g}{\partial x_{\iota_{r+1}}\partial x_{\iota_r}}.$$

Hence

$$f^{(r+1)}_{\iota_1\cdots\iota_{r-1}\iota_r\iota_{r+1}} = f^{(r+1)}_{\iota_1\cdots\iota_{r-1}\iota_{r+1}\iota_r}.$$

Differentiating now with respect to $x_{\iota_{r+2}}\cdots x_{\iota_n}$, in the order given, we get

$$f^{(n)}_{x_{\iota_1}\cdots x_{\iota_r}x_{\iota_{r+1}}\cdots x_{\iota_n}} = f^{(n)}_{x_{\iota_1}\cdots x_{\iota_{r-1}}x_{\iota_{r+1}}x_{\iota_r}\cdots x_{\iota_n}}.$$

Totally Differentiable Functions

423. 1. If the function $f(x)$ has a finite differential coefficient at $x = a$, we saw that

$$\Delta f = f'(a)h + \alpha h,$$

where h is an increment of x, and α is a function of h, such that

$$\lim_{h=0} \alpha = 0.$$

Under certain conditions, to be given later, an analogous theorem holds for functions of several variables. Let Δf be the increment that $f(x_1\cdots x_m)$ receives when we pass from the point $a = (a_1\cdots a_m)$ to the point $a + h = (a_1 + h_1\cdots a_m + h_m)$.

Here any of the h's may $= 0$. Let

$$\Delta f = f'_{x_1}(a)h_1 + \cdots + f'_{x_m}(a)h_m + \alpha_1 h_1 + \cdots + \alpha_m h_m,$$

where the α_r are functions of $h_1 \cdots h_m$, such that

$$\lim_{h=0} \alpha_1 = 0, \quad \cdots \lim_{h=0} \alpha_m = 0.$$

The function f is, in this case, said to be a *totally differentiable function at* a.

We call
$$df = f'_{x_1}(a)h_1 + \cdots + f'_{x_m}(a)h_m \tag*{(1}$$
the *total differential of* f *at* a.

Thus, when f is totally differentiable at a, Δf consists of two parts, viz.:
$$df \quad \text{and} \quad \alpha_1 h_1 + \cdots + \alpha_m h_m.$$

Here the α's in the second part have the limit 0 when $h \doteq 0$.

If we replace a by x and set $h_1 = dx_1, \cdots h_m = dx_m$, 1) becomes

$$df = f'_{x_1}(x)dx_1 + \cdots + f'_{x_m}(x)dx_m$$
$$= \frac{\partial f}{\partial x_1} dx_1 + \cdots + \frac{\partial f}{\partial x_m} dx_m.$$

424. 1. It is easy to give examples of functions which are *not totally differentiable* at every point.

Ex. 1.

Consider
$$f(x, y) = \sqrt{|xy|} = \sqrt[4]{x^2y^2}$$
at the origin.

Here
$$f'_x(0, 0) = 0, \qquad f'_y(0, 0) = 0.$$

Hence
$$df = 0 \tag*{(1}$$
at the origin.

Suppose now f were totally differentiable at the origin. Then the increment Δf would, on account of 1), have the form

$$\Delta f = \alpha h + \beta k, \tag*{(2}$$

where the limits of α and β are 0.

This is not possible.

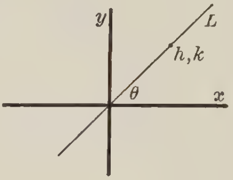
For, we have directly

$$\Delta f = f(h, k) - f(0, 0) = \sqrt{|hk|}. \tag{3}$$

From 2), 3) we have

$$\sqrt{|hk|} = \alpha h + \beta k. \tag{4}$$

To show now that the limits of α, β are not 0, let $h, k \doteq 0$, running over the line L, in the figure.



Then

$$h = \rho \cos\theta, \qquad k = \rho \sin\theta. \qquad \theta \textit{ constant.}$$

This in 4) gives

$$\rho\sqrt{\sin\theta\cos\theta} = \rho(\alpha\cos\theta + \beta\sin\theta),$$

or

$$\sqrt{\tfrac{1}{2}\sin 2\theta} = \alpha\cos\theta + \beta\sin\theta. \tag{5}$$

If now

$$\alpha \doteq 0, \qquad \beta \doteq 0,$$

the limit of the right side of 5) is 0; while the limit of the left side depends on θ. We are thus led to a contradiction.

2. Ex. 2.

$$f(xy) = \frac{xy}{\sqrt{x^2+y^2}}, \text{ for } x, y \text{ not the origin.}$$

$$= 0, \text{ for the origin.}$$

$$f_x'(0, 0) = \lim_{h=0} \frac{f(h, 0) - f(0, 0)}{h} = 0.$$

$$f_y'(0, 0) = 0.$$

Hence

$$df = 0,$$

at the origin. If now f were totally differentiable at the origin, we would have

$$\Delta f = \alpha h + \beta k,$$

or

$$r\cos\theta\sin\theta = r(\alpha\cos\theta + \beta\sin\theta).$$

Hence

$$\cos\theta\sin\theta = \alpha\cos\theta + \beta\sin\theta.$$

Letting now $h, k \doteq 0$, this gives, in the limit,

$$\cos\theta\sin\theta = 0,$$

which is absurd.

425. *Let $f(xy)$ be defined in the domain D of the point $P=(a, b)$. We suppose that:*

α) *f'_x exists in D,*
β) *f'_y exists at P,*
γ) *f'_x or f'_y is continuous at P.*

Then f is totally differentiable at P.

For,

$$\begin{aligned}\Delta f &= f(a+h, b+k) - f(a, b)\\ &= \{f(a+h, b+k) - f(a, b+k)\} + \{f(a, b+k) - f(ab)\}\\ &= \Delta_1 + \Delta_2. \end{aligned} \tag{1}$$

By α) we can apply the Law of the Mean to Δ_1, getting

$$\Delta_1 = hf'_x(c, b+k). \qquad a<c<a+h \text{ or } a+h<c<a. \tag{2}$$

By β) we have

$$\Delta_2 = k\{f'_y(a, b)+\beta\}, \tag{3}$$

where β is a function of k, such that

$$\lim_{k=0} \beta = 0. \tag{4}$$

From 1), 2), 3) we have

$$\Delta f = hf'_x(c, b+k) + k\{f'_y(ab)+\beta\}.$$

Set

$$\alpha = f'_x(c, b+k) - f'_x(a, b). \tag{5}$$

Then

$$\Delta f = hf'_x(ab) + kf'_y(a, b) + \alpha h + \beta k. \tag{6}$$

By γ), $\lim_{h, k=0} \alpha = 0.$ (7

Equations 6), 4), 7) show that f is totally differentiable at P.

426. 1. Under less general conditions we can generalize 425 as follows:

α) *Let $f(x_1 \cdots x_m)$ be defined over the domain D, of the point x; and have finite partial derivatives $f'_{x_1} \cdots f'_{x_m}$ in D.*

β) *Let f'_{x_κ} be a continuous function of $x_\kappa, x_{\kappa+1} \cdots x_m$.*

$$\kappa = 1, 2 \cdots m.$$

Then f is totally differentiable at x.

To fix the ideas, take $m = 3$. We have, setting for brevity,

$$\bar{x}_1 = x_1 + h_1, \quad \bar{x}_2 = x_2 + h_2, \quad \bar{x}_3 = x_3 + h_3;$$

$$\Delta f = f(\bar{x}_1\bar{x}_2\bar{x}_3) - f(x_1x_2x_3) = \{f(\bar{x}_1\bar{x}_2\bar{x}_3) - f(x_1\bar{x}_2\bar{x}_3)\}$$

$$+ \{f(x_1\bar{x}_2\bar{x}_3) - f(x_1x_2\bar{x}_3)\} + \{f(x_1x_2\bar{x}_3) - f(x_1x_2x_3)\}$$

$$= \Delta_1 + \Delta_2 + \Delta_3.$$

By virtue of α) we can apply the Law of the Mean to each of these Δ's, getting

$$\Delta_1 = h_1 f'_{x_1}(x_1 + \theta_1 h_1, \bar{x}_2, \bar{x}_3), \tag{1}$$

$$\Delta_2 = h_2 f'_{x_2}(x_1, x_2 + \theta_2 h_2, \bar{x}_3), \tag{2}$$

$$\Delta_3 = h_3 f'_{x_3}(x_1, x_2, x_3 + \theta_3 h_3). \tag{3}$$

Making use of β), we may write 1), 2), 3):

$$\Delta_1 = h_1 f'_{x_1}(x_1x_2x_3) + h_1\alpha_1, \tag{4}$$

$$\Delta_2 = h_2 f'_{x_2}(x_1x_2x_3) + h_2\alpha_2, \tag{5}$$

$$\Delta_3 = h_3 f'_{x_3}(x_1x_2x_3) + h_3\alpha_3; \tag{6}$$

and for $h \doteq 0$,

$$\lim \alpha_1 = 0, \quad \lim \alpha_2 = 0, \quad \lim \alpha_3 = 0.$$

Thus

$$\Delta f = df + \alpha_1 h_1 + \alpha_2 h_2 + \alpha_3 h_3.$$

Since the α's converge to 0, f is totally differentiable at the point x.

2. As a corollary of 1, we have:

Let $f(x_1 \cdots x_m)$ satisfy the conditions α), β) for a region R. Then f is totally differentiable at every point of R.

427. 1. *Let the partial derivative $f'_{x_\kappa}(x_1 \cdots x_m)$ be continuous in the region R. Then the difference quotient*

$$\frac{\Delta f}{\Delta x_\kappa} \doteq \frac{\partial f}{\partial x_\kappa} \text{ uniformly}$$

in any limited perfect domain D, in R.

For, by the Law of the Mean,

$$\frac{\Delta f}{\Delta x_\kappa} = f'_{x_\kappa}(x_1 \cdots x_\kappa + \theta \Delta x_\kappa \cdots x_m).$$

But f'_{x_κ}, being continuous in D, the function on the right converges uniformly to f_{x_κ}, by 352.

2. *Let the partial derivatives of the first order*

$$f'_{x_1},\ f'_{x_2},\ \cdots\ f'_{x_m}$$

be continuous in the region R. Then the α's in

$$\Delta f = df + \alpha_1 \Delta x_1 + \cdots + \alpha_m \Delta x_m$$

converge uniformly to 0, in any limited perfect domain D, in R.

For, referring to the proof of 426, we have

$$f'_{x_s}(x_1 \cdots x_{s-1},\ x_s + \theta_s \Delta x_s,\ \bar{x}_{s+1} \cdots \bar{x}_m) = f'_{x_s}(x_1 \cdots x_m) + \alpha_s.$$

Hence by 1, the α's are uniformly evanescent.

428. 1. *Let the first partial derivatives of $f(x_1 \cdots x_m)$ be continuous in the region R. Then f is continuous in R.*

We have to show that

$$\lim_{h=0} f(x_1 + h_1 \cdots x_m + h_m) = f(x_1 \cdots x_m);$$

or, what is the same,

$$\lim_{h=0} \Delta f = 0. \tag{1}$$

But, by 426, 2, f is totally differentiable in R; hence

$$\Delta f = df + \alpha_1 h_1 + \cdots + \alpha_m h_m. \tag{2}$$

As

$$\lim_{h=0} df = 0, \qquad \lim_{h=0} \alpha_1 = 0, \qquad \cdots \lim_{h=0} \alpha_m = 0;$$

passing to the limit in 2), we get 1).

2. As corollary, we have:

If all the partial derivatives of $f(x_1 \cdots x_m)$ of order n are continuous in the region R, then f, and all its partial derivatives of order $< n$, are also continuous in R.

429. 1. *At each point of a region R, let*

$$\Delta f = q_1 \Delta x_1 + \cdots + q_m \Delta x_m + \alpha_1 \Delta x_1 + \cdots + \alpha_m \Delta x_m; \tag{1}$$

where the q's are functions of x, and the α's are functions of x and Δx. Let

$$\lim_{\Delta x = 0} \alpha_\kappa = 0. \qquad \kappa = 1, 2, \cdots m. \tag{2}$$

Then f is totally differentiable in R, and

$$q_\kappa = \frac{\partial f}{\partial x_\kappa}. \tag{3}$$

For, let all the Δx_ι's be 0 except Δx_κ. Then

$$\frac{\Delta f}{\Delta x_\kappa} = q_\kappa + \alpha_\kappa.$$

Passing to the limit, we get 3). That f is totally differentiable follows now from 1), 2).

2. *Let*

$$df = \phi_1 dx_1 + \cdots + \phi_m dx_m$$

in R. Then

$$\phi_\kappa = \frac{\partial f}{\partial x_\kappa}. \qquad \kappa = 1, 2, \cdots m. \tag{4}$$

For, by definition,

$$df = \frac{\partial f}{\partial x_1} dx_1 + \cdots + \frac{\partial f}{\partial x_m} dx_m.$$

Let all the dx's, except dx_κ, be 0. Then

$$df = \phi_\kappa dx_\kappa = \frac{\partial f}{\partial x_\kappa} dx_\kappa;$$

or

$$\left(\phi_\kappa - \frac{\partial f}{\partial x_\kappa}\right) dx_\kappa = 0.$$

As $dx_\kappa \neq 0$, we have 4).

430. 1. *Let $f = f(u_1 \cdots u_n)$, and $u_\iota = g_\iota(x_1 \cdots x_m)$, $\iota = 1, 2, \cdots n$. Let the image of the region X be the region U. Let f be totally differentiable in U, and each g_ι be totally differentiable in X. Then f, considered as a function of the x's, is totally differentiable in X.*

Since the g's are totally differentiable,

$$\Delta u_\iota = \sum_\kappa \frac{\partial u_\iota}{\partial x_\kappa}\Delta x_\kappa + \Sigma_\kappa \alpha_{\iota\kappa}\Delta x_\kappa. \qquad \begin{matrix}\iota = 1, 2, \cdots n.\\ \kappa = 1, 2, \cdots m.\end{matrix} \tag{1}$$

Since $f(u_1 \cdots u_n)$ is totally differentiable,

$$\Delta f = \sum_\iota \frac{\partial f}{\partial u_\iota}\Delta u_\iota + \Sigma_\iota \beta_\iota \Delta u_\iota. \qquad \iota = 1, 2, \cdots n. \tag{2}$$

Replacing the values of Δu_ι, given by 1), in 2), we get

$$\begin{aligned}\Delta f = &\left(\frac{\partial f}{\partial u_1}\frac{\partial u_1}{\partial x_1} + \frac{\partial f}{\partial u_2}\frac{\partial u_2}{\partial x_1} + \cdots + \frac{\partial f}{\partial u_n}\frac{\partial u_n}{\partial x_1}\right)\Delta x_1\\ &+ \left(\frac{\partial f}{\partial u_1}\frac{\partial u_1}{\partial x_2} + \frac{\partial f}{\partial u_2}\frac{\partial u_2}{\partial x_2} + \cdots + \frac{\partial f}{\partial u_n}\frac{\partial u_n}{\partial x_2}\right)\Delta x_2\\ &+ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot\\ &+ \left(\frac{\partial f}{\partial u_1}\frac{\partial u_1}{\partial x_m} + \frac{\partial f}{\partial u_2}\frac{\partial u_2}{\partial x_m} + \cdots + \frac{\partial f}{\partial u_n}\frac{\partial u_n}{\partial x_m}\right)\Delta x_m\\ &+ \gamma_1 \Delta x_1 + \cdots + \gamma_m \Delta x_m;\end{aligned} \tag{3}$$

where

$$\lim \gamma_1 = 0, \cdots \lim \gamma_m = 0, \tag{4}$$

when

$$\Delta x_1, \cdots \Delta x_m \doteq 0.$$

Then, by 429, 1), f, considered as a function of the x's, is totally differentiable; and

$$\begin{aligned}\frac{\partial f}{\partial x_1} &= \frac{\partial f}{\partial u_1}\frac{\partial u_1}{\partial x_1} + \cdots + \frac{\partial f}{\partial u_n}\frac{\partial u_n}{\partial x_1} = \sum_\kappa \frac{\partial f}{\partial u_\kappa}\frac{\partial u_\kappa}{\partial x_1}\\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot\\ \frac{\partial f}{\partial x_m} &= \frac{\partial f}{\partial u_1}\frac{\partial u_1}{\partial x_m} + \cdots + \frac{\partial f}{\partial u_n}\frac{\partial u_n}{\partial x_m} = \sum_\kappa \frac{\partial f}{\partial u_\kappa}\frac{\partial u_\kappa}{\partial x_m}.\end{aligned} \tag{5}$$

Also,

$$df = \sum_\iota \frac{\partial f}{\partial x_\iota}dx_\iota = \sum_\iota dx_\iota \sum_\kappa \frac{\partial f}{\partial u_\kappa}\frac{\partial u_\kappa}{\partial x_\iota} = \sum_{\iota,\kappa} \frac{\partial f}{\partial u_\kappa}\frac{\partial u_\kappa}{\partial x_\iota}dx_\iota. \tag{6}$$

2. We have, f being considered as a function of the x's,

$$df = \sum_{\iota} dx_{\iota} \sum_{\kappa} \frac{\partial f}{\partial u_{\kappa}} \frac{\partial u_{\kappa}}{\partial x_{\iota}} = \sum_{\kappa} \frac{\partial f}{\partial u_{\kappa}} \sum_{\iota} \frac{\partial u_{\kappa}}{\partial x_{\iota}} dx_{\iota};$$

or

$$df = \sum_{\kappa} \frac{\partial f}{\partial u_{\kappa}} du_{\kappa}.$$

Thus to find df, f considered as a function of the x's, we may first find df, considered as a function of the u's, and in the result, replace the du's by their values in the x's.

3. As a corollary of 1, we have:

Let

$$\frac{\partial f}{\partial u_1}, \cdots \frac{\partial f}{\partial u_n}$$

be continuous in U. Let

$$\frac{\partial u_{\iota}}{\partial x_1}, \cdots \frac{\partial u_{\iota}}{\partial x_m} \qquad \iota = 1, 2, \cdots n.$$

be continuous in X. Then f, considered as a function of the x's, is totally differentiable in X.

For, by 426, 2, f considered as a function of the u's, and the u's considered as functions of the x's, are all totally differentiable. Hence, by 1, f considered as a function of the x's, is totally differentiable.

Some Properties of Differentials. Higher Differentials

431. In this section we shall suppose the total differentials which occur exist in a certain region R, in which $x = (x_1 \cdots x_m)$ ranges.

1. *Let*

$$F = c_1 f_1 + \cdots + c_n f_n. \qquad \textit{c's constants.}$$

Then

$$dF = c_1 df_1 + \cdots + c_n df_n.$$

For,

$$dF = \sum_{\iota} \frac{\partial F}{\partial x_{\iota}} dx_{\iota} = \sum_{\iota\kappa} c_{\kappa} \frac{\partial f_{\kappa}}{\partial x_{\iota}} dx_{\iota}$$

$$= \sum_{\kappa} c_{\kappa} \sum_{\iota} \frac{\partial f_{\kappa}}{\partial x_{\iota}} dx_{\iota} = \sum c_{\kappa} df_{\kappa}.$$

2. *Let*

$$F = fg.$$

Then

$$dF = fdg + gdf.$$

For,

$$dF = \sum \frac{\partial (fg)}{\partial x_\iota} dx_\iota = \sum f \frac{\partial g}{\partial x_\iota} dx_\iota + \sum g \frac{\partial f}{\partial x_\iota} dx_\iota$$

$$= f \sum \frac{\partial g}{\partial x_\iota} dx_\iota + g \sum \frac{\partial f}{\partial x_\iota} dx_\iota = fdg + gdf.$$

3. *Let*

$$F = \frac{f}{g}. \qquad g \neq 0 \text{ in } R.$$

Then

$$dF = \frac{gdf - fdg}{g^2}.$$

For,

$$dF = \sum \frac{\partial F}{\partial x_\iota} dx_\iota = \sum \frac{g \dfrac{\partial f}{\partial x_\iota} - f \dfrac{\partial g}{\partial x_\iota}}{g^2} dx_\iota$$

$$= \frac{1}{g} \sum \frac{\partial f}{\partial x_\iota} dx_\iota - \frac{f}{g^2} \sum \frac{\partial g}{\partial x_\iota} dx_\iota$$

$$= \frac{df}{g} - \frac{fdg}{g^2} = \frac{gdf - fdg}{g^2}.$$

432. The partial derivatives involved, being supposed continuous in a certain region R, let us form the expressions

$$d^2 f = \sum_{\iota, \kappa} \frac{\partial^2 f}{\partial x_\iota \partial x_\kappa} dx_\iota dx_\kappa, \qquad \iota, \kappa = 1, 2, \cdots m.$$

$$d^3 f = \sum_{\iota, \jmath, \kappa} \frac{\partial^3 f}{\partial x_\iota \partial x_\jmath \partial x_\kappa} dx_\iota dx_\jmath dx_\kappa, \qquad \iota, \jmath, \kappa = 1, 2, \cdots m.$$

. . . . . .

They are called the *second*, *third*, $\cdots$ *differentials* of $f(x_1 \cdots x_m)$, respectively, in R.

We notice:

$$d^2f = \sum_\iota dx_\iota \sum_\kappa \frac{\partial^2 f}{\partial x_\iota \partial x_\kappa} dx_\kappa$$

$$= \sum_\iota dx_\iota \sum_\kappa \frac{\partial}{\partial x_\kappa}\left(\frac{\partial f}{\partial x_\iota}\right) dx_\kappa$$

$$= \sum_\iota dx_\iota \cdot d\left(\frac{\partial f}{\partial x_\iota}\right)$$

$$= d \sum_\iota \frac{\partial f}{\partial x_\iota} dx_\iota ; \qquad (1$$

since dx_ι acts as a constant with respect to the symbol d.

Then 1) gives

$$d^2 f = d \cdot df.$$

In the same way we find

$$d^n f = d \cdot d^{n-1} f = d^2 \cdot d^{n-2} f \cdots$$

433. 1. Let $w = f(u_1 \cdots u_n)$, while $u_1 \cdots u_n$ are functions of $x_1 \cdots x_m$. Let w, when considered as a function of the x's, be denoted by $F(x_1 \cdots x_m)$. Let finally all derivatives involved be continuous. Then

$$dF = \sum_\iota \frac{\partial f}{\partial u_\iota} du_\iota . \qquad (1$$

$$d^2 F = \sum_\iota du_\iota \, d \cdot \frac{\partial f}{\partial u_\iota} + \sum_\iota \frac{\partial f}{\partial u_\iota} d^2 u_\iota$$

$$= \sum_\iota du_\iota \sum_\kappa \frac{\partial^2 f}{\partial u_\iota \partial u_\kappa} du_\kappa + \sum_\iota \frac{\partial f}{\partial u_\iota} d^2 u_\iota$$

$$= d^2 f + \sum_\iota \frac{\partial f}{\partial u_\iota} d^2 u_\iota . \qquad (2$$

$$d^3F = \sum_{\iota\kappa} du_\iota\, du_\kappa\, d\cdot\frac{\partial^2 f}{\partial u_\iota\,\partial u_\kappa} + \sum_{\iota\kappa}\frac{\partial^2 f}{\partial u_\iota\,\partial u_\kappa}\,du_\iota\, d^2u_\kappa$$

$$+\sum_{\iota\kappa}\frac{\partial^2 f}{\partial u_\iota\,\partial u_\kappa}\,du_\kappa\, d^2u_\iota + \sum_\iota d^2u_\iota\, d\cdot\frac{\partial f}{\partial u_\iota}$$

$$+\sum_\iota \frac{\partial f}{\partial u_\iota}\,d^3u_\iota$$

$$=\sum_{\iota\kappa\lambda}\frac{\partial^3 f}{\partial u_\iota\,\partial u_\kappa\,\partial u_\lambda}\,du_\iota\,du_\kappa\,du_\lambda + 3\sum_{\iota,\kappa}\frac{\partial^2 f}{\partial u_\iota\,\partial u_\kappa}\,d^2u_\iota\,du_\kappa + \sum_\iota\frac{\partial f}{\partial u_\iota}\,d^3u_\iota$$

$$= d^3f + 3\sum_{\iota,\kappa}\frac{\partial^2 f}{\partial u_\iota\,\partial u_\kappa}\,d^2u_\iota\,du_\kappa + \sum_\iota\frac{\partial f}{\partial u_\iota}\,d^3u_\iota. \qquad (3$$

In the same way the higher differentials $d^4F \cdots$ can be calculated.

2. In case

$$d^2u_\iota = 0,\ \ d^3u_\iota = 0,\ \cdots$$

we have

$$dF = df,\ \ d^2F = d^2f,\ \ d^3F = d^3f,\ \cdots$$

434. *Let all the partial derivatives of $f(x_1 \cdots x_m)$ of order n be continuous in the domain D of the point x.*

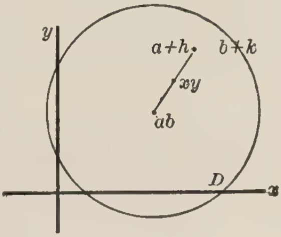
Then, if $x + h$ lies in D,

$$f(x_1 + h_1 \cdots x_m + h_m) = f(x_1 \cdots x_m) + \frac{1}{1!}\,df(x_1 \cdots x_m) + \frac{1}{2!}\,d^2f(x_1 \cdots x_m)$$

$$+ \cdots + \frac{1}{n!}\,d^nf(x_1 + \theta h_1 \cdots x_m + \theta h_m); \qquad (1$$

setting $dx_\iota = h_\iota$, *and* $0 < \theta < 1$.

The expression on the right of 1) is called *Taylor's development of f in finite form.*



For simplicity we shall suppose $m = 2$. The reasoning in the general case is precisely the same.

To avoid writing indices, we shall call the variables x, y.

Let x, y be any point on the line L joining the points a, b and $a+h$, $b+k$. Then

$$x=a+uh, \quad y=b+uk. \qquad 0 \leq u \leq 1.$$

Also let
$$f(x, y)=f(a+uh, \quad b+uk)=g(u).$$

Then, when u runs over the interval $\mathfrak{A}=(0, 1)$, the point xy runs over the interval on L between a, b and $a+h$, $b+k$.

By 433, 1, we have

$$g'(u)=\frac{\partial f}{\partial x}h+\frac{\partial f}{\partial y}k=df(x, y),$$

$$g''(u)=d^2f(x, y), \quad \cdots g^{(n)}(u)=d^nf(x, y).$$

Thus $g(u)$ and its first n derivatives are continuous functions of u in $\mathfrak{A}$.

Applying 409, we have

$$g(u)=g(0)+\frac{u}{1!}g'(0)+\frac{u^2}{2!}g''(0)+\cdots+\frac{u^n}{n!}g^{(n)}(\theta u).$$

$$0<\theta<1.$$

Setting here $u=1$, and observing that

$$g(1)=f(a+h, b+k), \quad g(0)=f(a, b),$$

$$g'(0)=df(a, b), \quad g''(0)=d^2f(a, b), \cdots$$

$$g^{(n)}(\theta u)=d^nf(a+\theta h, b+\theta k),$$

we get

$$f(a+h, b+k)=f(a, b)+\frac{1}{1!}df(a, b)+\frac{1}{2!}d^2f(a, b)+\cdots$$
$$+\frac{1}{n-1!}d^{n-1}f(a, b)+\frac{1}{n!}d^nf(a+\theta h, b+\theta k).$$

435. In Taylor's development of a function $f(x)$ of a single variable [409], we have only assumed that $f^{(n)}(x)$ is *finite* within $\mathfrak{A}$; whereas, in the corresponding development of a function of several variables, we have assumed [434] that all the partial derivatives of order n are *continuous* in $D(a)$, in order to use 430.

It is interesting to note that the development may not hold if these derivatives are not continuous.

Consider the function

$$f(xy) = \sqrt{|xy|},$$

employed in 424, 1.

We have

$$f'_x = \frac{1}{2}\frac{xy^2}{\sqrt{|xy|^3}}, \quad f'_y = \frac{1}{2}\frac{x^2y}{\sqrt{|xy|^3}}; \qquad x, y \neq 0.$$

$$f'_x(x, 0) = 0, \ f'_y(0, y) = 0.$$

The derivatives of the first order are thus continuous, except at the origin.

Let $P = (x, x)$, $Q = (x+h, x+h)$ be two points on the line $y = x$, which we call L.

If now Taylor's development were true in a domain about a, in which the nth partial derivatives were finite, we could write, taking here $n = 1$,

$$f(x+h, x+h) = f(x, x) + h\{f'_x(\xi, \xi) + f'_y(\xi, \xi)\}, \tag{1}$$

where (ξ, ξ) is a point on L between P, Q.

This formula should be valid for all x, h. But in the present case

$$f'_x(\xi, \xi) = f'_y(\xi, \xi) = \tfrac{1}{2} \operatorname{sgn} \xi.$$

Thus 1) gives

$$|x+h| = |x| + h \operatorname{sgn} \xi. \tag{2}$$

That this result is false is easily seen.

For example, let

$$x = -1, \ h = 5.$$

Then 2) gives

$$4 = 1 \pm 5, \qquad \text{if } \xi \neq 0$$

$$= 1, \qquad \text{if } \xi = 0.$$

CHAPTER IX

IMPLICIT FUNCTIONS

436. 1. Let

$$F(x_1 \cdots x_m, u) = 0 \tag{1}$$

be a relation between the $m+1$ variables $x_1 \cdots x_m, u$. Let

$$x_1 = a_1, \cdots x_m = a_m$$

be a set of values such that the equation

$$F(a_1 \cdots a_m, u) = 0 \tag{2}$$

is satisfied for at least one value of u; *i.e.* the equation 2) in u admits at least one root. Let D be the aggregate of the points $x = (x_1 \cdots x_m)$ for which 1) has at least one root u. We may consider u as a function of the x's, $u = \phi(x_1 \cdots x_m)$ defined over D, where $\phi(x_1 \cdots x_m)$ has assigned to it at the point x, the roots u of 1) at this point.

We say u is the *implicit function* defined by 1). It is in general a many valued function.

EXAMPLES

1. Let

$$y = f(x) \tag{3}$$

be defined over a domain D. Let E be the image of D. Then 3) defines an inverse function

$$x = g(y),$$

defined over E, by 217. This same function may be considered as an implicit function, defined by

$$y - f(x) = F(x, y) = 0.$$

2. Let $f(x) = 1$ for every x in $D = (01)$. If we set

$$y = f(x),$$

the image E of D is the single point $y = 1$. The inverse function

$$x = g(y),$$

is defined only for $y = 1$; at this point g takes on all values between 0 and 1.

3. Let $F = 0$ be the relation

$$x^2 + y^2 + z^2 - r^2 = 0, \qquad r \neq 0. \tag{4}$$

At each point of the domain D,

$$x^2 + y^2 \overline{\overline{<}} r^2,$$

the equation 4) admits one, and in general, two values of z. The equation 4) therefore defines z as a two-valued implicit function u of x, y, over the domain D.

4.
$$x^2 + y^2 + z^2 = 0. \tag{5}$$

In this case there is only *one* set of values, viz. $x = y = z = 0$ satisfying 5). Thus z is defined only for a single point, viz. $x = y = 0$. At this point, $z = 0$.

5.
$$x^2 + y^2 + z^2 + r^2 = 0. \qquad r \neq 0. \tag{6}$$

This equation is satisfied for no set of values of x, y, z. The equation 6), therefore, does not define any function z of x, y.

6.
$$\sin^2 u + \cos^2 u - \frac{x}{y} = 0. \tag{7}$$

This equation admits no solution except for points on the line

$$y = x.$$

For all points on this line, the origin excepted, the equation 7) is satisfied for *any* value of u in $\mathfrak{R}$.

2. More generally, let

$$\begin{aligned} F_1(x_1 \cdots x_m, u_1 \cdots u_p) &= 0 \\ \cdot \quad \cdot \quad \cdot \quad &\cdot \quad \cdot \\ F_p(x_1 \cdots x_m, u_1 \cdots u_p) &= 0 \end{aligned} \tag{S}$$

be a system of p relations between the $m + p$ variables x, u. Let D be the aggregate of points $x = (x_1 \cdots x_m)$, for which the system S is satisfied for at least one set of values of $u_1 \cdots u_p$. We may consider the u's as functions of the x's.

$$\begin{aligned} u_1 &= \phi_1(x_1 \cdots x_m) \\ \cdot \quad &\cdot \quad \cdot \quad \cdot \\ u_p &= \phi_p(x_1 \cdots x_m), \end{aligned}$$

where the ϕ's have assigned to them at the point x, the values of the roots $u_1 \cdots u_p$ at this point. We say $u_1 \cdots u_p$ is a *system of implicit functions* defined by the system S. These functions are, in general, many valued.

3. Suppose we know that a set of values

$$x_1 = a_1, \cdots x_m = a_m, u_1 = b_1, \cdots u_p = b_p$$

satisfies the system S. Let us call the set of values $u_1 = b_1 \cdots u_p = b_p$ *initial values.*

We wish to show now that under certain conditions, the system S defines over a region R a set of p one-valued *continuous* functions $u_1 \cdots u_p$ in the variables $x_1 \cdots x_m$, satisfying S for every point of R, and taking on the above initial values at the point $x = a$. Furthermore there is only one such system of functions.

The method employed is due to Goursat, *Bull. Soc. Math. de France*, vol. 31 (1903), p. 184. It rests on a principle, having many applications in analysis, known as the *Method of Successive Approximation.*

437. 1. Let us first consider only two variables. The method employed for this simple case is readily extended to the most general case. We begin by establishing the fundamental

Lemma. Let $f(x, u)$ *be continuous, and* $\frac{\partial f}{\partial u}$ *exist in the domain* D, *defined by*

$\mathfrak{A}$; $$|x - a| \leq \sigma,$$

$\mathfrak{B}$; $$|u - b| \leq \tau.$$

Let f *vanish at* a, b. *Let* θ *be an arbitrary positive number* < 1, *such that*

$$\left|\frac{\partial f}{\partial u}\right| < \theta, \qquad \text{in } D, \tag{1}$$

while

$$|f(x, b)| < \tau(1 - \theta) = \eta < \tau, \text{ in } \mathfrak{A}. \tag{2}$$

Then

$$u - b = f(x, u) \tag{3}$$

admits one and only one solution

$$U = \phi(x), \qquad \text{in } \mathfrak{A}.$$

which is continuous at a, *and takes on the initial value* $u = b$ *at* $x = a$.

The function ϕ is continuous in $\mathfrak{A}$, and remains in $\mathfrak{B}$ while x runs over $\mathfrak{A}$.

We set

$$u_1 - b = f(x, b),\ u_2 - b = f(x, u_1),\ u_3 - b = f(x, u_2), \cdots$$

Then all these u's fall in $\mathfrak{B}$.

For, by 2), u_1 falls in $\mathfrak{B}$. Let us admit that u_{r-1} falls in $\mathfrak{B}$, and show that u_r also falls in $\mathfrak{B}$.

In fact, by the Law of the Mean,

$$\begin{aligned} u_r - u_1 &= (u_r - b) - (u_1 - b) \qquad (4 \\ &= f(x, u_{r-1}) - f(x, b) \\ &= (u_{r-1} - b)\frac{\partial f(x, u')}{\partial u} \end{aligned}$$

Hence, by 1),

$$\begin{aligned} |u_r - u_1| &< \theta |u_{r-1} - b| \qquad (5 \\ &< \theta\tau, \qquad (6 \end{aligned}$$

since, by hypothesis, u_{r-1} falls in $\mathfrak{B}$.

Thus, from $u_r - b = (u_r - u_1) + (u_1 - b)$ and 6), we have

$$|u_r - b| < |u_1 - b| + \theta\tau.$$

But

$$|u_1 - b| = |f(x, b)| < (1 - \theta)\tau, \text{ by 2).}$$

Hence,

$$|u_r - b| < \tau; \qquad (7$$

i.e. all the u's fall in $\mathfrak{B}$.

We show now that for each x in $\mathfrak{A}$,

$$U = \lim u_n = \phi(x)$$

is finite. To this end we show

$$\epsilon > 0, \quad m, \quad |u_n - u_m| < \epsilon, \quad n > m. \qquad (8$$

For, in the same way that we established 5), we can show that

$$|u_r - u_{r-1}| < \theta |u_{r-1} - u_{r-2}|. \qquad (9$$

Thus, we get

$$
\begin{aligned}
&|u_2 - u_1| < \theta |u_1 - b| < \theta\eta, \\
&|u_3 - u_2| < \theta |u_2 - u_1| < \theta^2\eta, \qquad\qquad (10 \\
&|u_4 - u_3| < \theta |u_3 - u_2| < \theta^3\eta, \text{ etc.}
\end{aligned}
$$

Hence

$$
\begin{aligned}
|u_n - u_m| &< \eta\theta^m(1 + \theta + \cdots + \theta^{n-m-1}) \\
&< \frac{\eta\theta^m}{1-\theta}, \qquad \text{since } 0 < \theta < 1 \\
&< \epsilon,
\end{aligned}
$$

if m is taken sufficiently large.

Thus the relation 8) is established.

Furthermore, the above reasoning shows that one and the same m suffices wherever x is taken in $\mathfrak{A}$. Thus u_n *converges uniformly to* U *in* $\mathfrak{A}$.

Finally, by virtue of 7), U falls in $\mathfrak{B}$.

The function U satisfies 3) in $\mathfrak{A}$.

For, in

$$u_n - b = f(x, u_{n-1})$$

let $n \doteq \infty$. Since f is continuous, we get in the limit

$$U - b = f(x, U).$$

We show now that $U = \phi(x)$ is continuous in $\mathfrak{A}$. For, since u_n converges uniformly to $\phi(x)$ in $\mathfrak{A}$, we have

$$\phi(x+h) = u_n(x+h) + \epsilon', \qquad |\epsilon'| < \frac{\epsilon}{3}.$$

$$\phi(x) = u_n(x) + \epsilon'', \qquad |\epsilon''| < \frac{\epsilon}{3}.$$

if n is taken large enough.

But $u_1, u_2, \cdots$ are continuous functions of x, since f is continuous. Thus, for sufficiently small δ,

$$|u_n(x+h) - u_n(x)| < \frac{\epsilon}{3}$$

for $|h| < \delta$ and $x + h$ in $\mathfrak{A}$.

Hence

$$|\phi(x+h)-\phi(x)|<\epsilon.$$

We show now that U is the only root of 3) which is continuous at a and takes on the initial value b at a.

For, let $V=\psi(x)$ be such a solution.

If x is taken sufficiently near a, V falls in $\mathfrak{B}$.

Then from

$$V-b=f(x, V),$$

$$u_n-b=f(x, u_{n-1}),$$

we have, by the Law of the Mean,

$$|V-u_n|=|f(x, V)-f(x, u_{n-1})|$$

$$=|V-u_{n-1}|\left|\frac{\partial f(x, u')}{\partial u}\right|$$

$$<\theta|V-u_{n-1}|, \quad \text{by 1)},$$

$$<\theta^{n-1}|V-u_1|.$$

Hence, passing to the limit, $n=\infty$,

$$V-U=0, \quad \text{for all points of } \mathfrak{A}.$$

2. As corollary of 1, we have:

Let

$$f(x, u), \ \frac{\partial f}{\partial u}$$

be continuous in the domain of the point (a, b), *and vanish at that point.*

Then

$$u-b=f(x, u)$$

admits a unique solution

$$u=\phi(x),$$

which is continuous in the domain of $x=a$, *and has the initial value* $u=b$ *at* $x=a$.

438. 1. By means of the preceding lemma, we can now prove the theorem:

Let $F(x, u)$ be continuous, and F'_u exist in the domain D, defined by

$$\mathfrak{A}; \qquad |x - a| \leq \sigma,$$

$$\mathfrak{B}; \qquad |u - b| \leq \tau.$$

At a, b let $F = 0$, while $F'_u \neq 0$.

Let θ be an arbitrary positive number < 1, such that

$$\left|1 - \frac{F'_u(x, u)}{F'_u(a, b)}\right| < \theta, \qquad \text{in } D; \tag{1}$$

while

$$\left|\frac{F(x, b)}{F'_u(a, b)}\right| < (1 - \theta)\tau, \qquad \text{in } \mathfrak{A}. \tag{2}$$

Then the equation

$$F(x, u) = 0 \tag{3}$$

is satisfied by a one-valued continuous function

$$u = \phi(x), \qquad \text{in } \mathfrak{A};$$

having the initial value b at $x = a$, and which remains in $\mathfrak{B}$ while x is in $\mathfrak{A}$.

Furthermore, 3) *admits no other solution which is continuous at a, and has the initial value b at a.*

For, consider the equation

$$u - b = u - b - \frac{F(x, u)}{F'_u(a, b)} = f(x, u). \tag{4}$$

Evidently this is equivalent to 3); *i.e.* every function u which satisfies 3) satisfies 4), and conversely.

Here

$$\frac{\partial f}{\partial u} = 1 - \frac{F'_u(x, u)}{F'_u(a, b)}.$$

Hence f is continuous, and f'_u exists in D; also f vanishes at a, b.

Furthermore, from 1)

$$\left|\frac{\partial f}{\partial u}\right| < \theta, \qquad \text{in } D.$$

while from 2),

$$|f(x,\, b)| < (1-\theta)\tau, \qquad \text{in } \mathfrak{A}.$$

Thus f and f'_u satisfy all the conditions of the lemma in 437, and the theorem follows at once.

2. The reader should remark that the preceding theorem makes no assumption regarding F'_x. This may not even exist.

For example, let

$$x \sin\frac{1}{x} = 0, \text{ for } x = 0.$$

Consider

$$F(x,\, u) = u^3 - x\sin\frac{1}{x} = 0. \tag{5}$$

Here F'_x does not exist at $x = 0$, $u = 0$. However, the equation 5) defines a continuous one-valued function, which takes on the initial value $u = 0$ for $x = 0$; viz.,

$$u = \sqrt[3]{x \sin\frac{1}{x}}.$$

3. As corollary of 1 we have:

In the domain of the point a, b, let

$$F(x,\, u),\ F'_u(x,\, u)$$

be continuous. At the point a, b, let

$$F = 0,\ F'_u \neq 0.$$

Then the equation $F(x,\, u) = 0$ admits a unique solution

$$u = \phi(x),$$

which is continuous in the domain of the point $x = a$, and has the initial value $u = b$, at this point.

439. We have seen in 438 that

$$F(x, u) = 0 \qquad (1$$

is satisfied by a continuous function

$$u = \phi_1(x)$$

in a certain interval $(a - \sigma, a + \sigma) = (A, B)$; and that there is only one such function which $= b$, when $x = a$. In general this is true not only for the interval (A, B) determined by the theorem 438, but for a larger interval (C, D), containing (A, B). For, let a_1 be a point near one of the end points of (A, B). Let $b_1 = \phi_1(a_1)$. Let us replace a, b in the theorem of 438 by a_1, b_1. Then the conditions of this theorem are satisfied for a certain interval (A_1, B_1), about a_1; to which corresponds a continuous function

$$u = \phi_2(x),$$

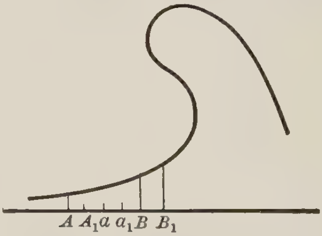
determined by the condition that $u = b_1$, for $x = a_1$. The interval (A_1, B_1) will in general extend beyond (A, B). In the interval (A_1, B) which the two intervals (A, B), (A_1, B_1) have in common, the two functions

$$\phi_1(u), \quad \phi_2(u)$$

are equal. Let us define a function

$$u = \phi(x) = \phi_1(x), \text{ in } (A, B),$$
$$= \phi_2(x), \text{ in } (A_1, B_1).$$

Then the equation 1) is satisfied by this function in (A, B_1), and it is uniquely determined by the fact that it is continuous in (A, B_1) and has the value $u = b$, for $x = a$. In this way we can continue extending on the right, and on the left, the original interval, until we are blocked by certain points beyond which we cannot go. Such points may arise when $F(x, u)$ ceases to be continuous, or when $F_u{}' = 0$.



440. 1. We proceed now to extend the theorem of 438 to embrace the system S of 436.

To this end we generalize the lemma of 437 as follows:

Lemma. Let

$$f_1(x_1 \cdots x_m u_1 \cdots u_p) \cdots f_p(x_1 \cdots x_m u_1 \cdots u_p),$$

and

$$\frac{\partial f_\iota}{\partial u_\kappa} \qquad \iota, \kappa = 1, 2, \cdots p.$$

be continuous in the domain D defined by

$$\mathfrak{A}; \qquad |x_1 - a_1| \le \sigma \cdots |x_m - a_m| \le \sigma,$$

$$\mathfrak{B}: \qquad |u_1 - b_1| \le \tau \cdots |u_p - b_p| \le \tau,$$

and let $f_1, \cdots f_p$ vanish at the point $(a_1 \cdots a_m b_1 \cdots b_p)$.

Let θ be an arbitrary positive number < 1, such that

$$\left|\frac{\partial f_\iota}{\partial u_\kappa}\right| < \frac{\theta}{p}, \qquad \iota, \kappa = 1, 2, \cdots p, \quad in\ D. \tag{1}$$

while

$$|f_\iota(x_1 \cdots x_m b_1 \cdots b_p)| < \tau(1-\theta) = \eta, \quad in\ \mathfrak{A}. \qquad \iota = 1, 2, \cdots p. \tag{2}$$

Then the equations

$$u_1 - b_1 = f_1(x_1 \cdots u_p) \quad \cdots \quad u_p - b_p = f_p(x_1 \cdots u_p)$$

admit one, and only one, set of solutions

$$U_1 = \phi_1(x_1 \cdots x_m) \quad \cdots \quad U_p = \phi_p(x_1 \cdots x_m)$$

in $\mathfrak{A}$, which are continuous at a, and take on the initial values $b_1 \cdots b_p$, at $x = a$.

The functions ϕ are continuous in $\mathfrak{A}$, and remain in $\mathfrak{B}$, as x runs over $\mathfrak{A}$.

We set

$$\begin{aligned}
u_{11} - b_1 &= f_1(x_1 \cdots x_m b_1 \cdots b_p) \\
&\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
u_{p1} - b_p &= f_p(x_1 \cdots x_m b_1 \cdots b_p) \\
u_{12} - b_1 &= f_1(x_1 \cdots x_m u_{11} \cdots u_{p1}) \\
&\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
u_{p2} - b_p &= f_p(x_1 \cdots x_m u_{11} \cdots u_{p1})
\end{aligned}$$

etc.

We show now that all these u's lie in $\mathfrak{B}$.

For, by 2), $u_{11} \cdots u_{p1}$ are in $\mathfrak{B}$. Let us assume now that $u_{1,r-1} \cdots u_{p,r-1}$ lie in $\mathfrak{B}$, and show that $u_{1r} \cdots u_{pr}$ also lie in $\mathfrak{B}$. By the Law of the Mean,

$$u_{\iota r} - u_{\iota 1} = (u_{1,r-1} - b_1)\frac{\partial f_\iota}{\partial u_1} + \cdots + (u_{p,r-1} - b_p)\frac{\partial f_\iota}{\partial u_p},$$

the arguments of these derivatives lying in D. Hence, by 1),

$$|u_{\iota r} - u_{\iota 1}| < \frac{\theta}{p}\{|u_{1,r-1} - b_1| + \cdots + |u_{p,r-1} - b_p|\} \tag{3}$$

$$< \theta\tau.$$

Thus, as

$$u_{\iota r} - b_\iota = (u_{\iota r} - u_{\iota 1}) + (u_{\iota 1} - b_\iota),$$

we have

$$|u_{\iota r} - b_\iota| < |u_{\iota 1} - b_\iota| + \theta\tau, \qquad \iota = 1, 2, \cdots p.$$

or using 2),

$$|u_{\iota r} - b_\iota| < \tau, \quad \iota = 1, 2, \cdots p; \ r = 2, 3, \cdots$$

which was to be shown.

We show now that for each x in $\mathfrak{A}$,

$$U_\iota = \lim_{n=\infty} u_{\iota, n} = \phi_\iota(x_1 \cdots x_m)$$

is finite.

To this end we show that

$$\epsilon > 0, \quad m, \quad |u_{\iota n} - u_{\iota m}| < \epsilon. \qquad n > m. \tag{4}$$

$$\iota = 1, 2, \cdots p.$$

For, as in 3),

$$|u_{\iota r} - u_{\iota, r-1}| < \frac{\theta}{p}\{|u_{1,r-1} - u_{1,r-2}| + \cdots |u_{p,r-1} - u_{p,r-2}|\}. \tag{5}$$

Then

$$|u_{\iota 2} - u_{\iota, 1}| < \theta\eta, \quad \text{by 2) and 3)},$$

$$|u_{\iota 3} - u_{\iota, 2}| < \theta^2\eta, \quad \text{by 5)},$$

$$|u_{\iota, 4} - u_{\iota, 3}| < \theta^3\eta, \text{ etc.}$$

These relations are analogous to the relations 10) in 437. The rest of the demonstration can now be conducted as in 437 to establish not only the relation 4), but the remainder of the theorem in hand.

2. We can state 1 in a form less explicit, but easier to remember, as follows:

Let

$$f_1(x_1 \cdots x_m, u_1 \cdots u_p) \quad \cdots \quad f_p(x_1 \cdots x_m, u_1 \cdots u_p),$$

and

$$\frac{\partial f_\iota}{\partial u_\kappa} \qquad \iota, \kappa = 1, 2, \cdots p.$$

be continuous in the domain of the point $a_1 \cdots a_m b_1 \cdots b_p$.

Let these $p^2 + p$ *functions vanish at this point.*

Then the system of equations

$$u_1 - b_1 = f_1(x_1 \cdots u_p) \quad \cdots \quad u_p - b_p = f_p(x_1 \cdots u_p)$$

admits a unique system of solutions

$$u_1 = \phi_1(x_1 \cdots x_m) \quad \cdots \quad u_p = \phi_p(x_1 \cdots x_m),$$

which is continuous in the domain of the point $x_1 = a_1 \cdots x_m = a_m$, *and takes on the initial set of values* $u_1 = b_1 \cdots u_p = b_p$.

441. We can now generalize 438 as follows:

Let

$$F_1(x_1 \cdots x_m u_1 \cdots u_p) \quad \cdots \quad F_p(x_1 \cdots x_m u_1 \cdots u_p),$$

and

$$\frac{\partial F_\iota}{\partial u_\kappa} \qquad \iota, \kappa = 1, 2, \cdots p. \tag{1}$$

be continuous in the domain D *of the point*

$$Q; \qquad x_1 = a_1 \cdots x_m = a_m, \; u_1 = b_1 \cdots u_p = b_p.$$

Let $F_1 \cdots F_p$ *vanish at* Q, *while the derivatives* 1) *have the values*

$$d_{\iota\kappa} \qquad \text{at } Q.$$

Let

$$\Delta = \begin{vmatrix} d_{11} \cdots d_{1p} \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ d_{p1} \cdots d_{pp} \end{vmatrix} \neq 0.$$

Then the system of equations

$$F_1 = 0 \cdots F_p = 0 \tag{S}$$

is satisfied by a set of functions

$$u_1 = \phi_1(x_1 \cdots x_m)$$

$$\cdot \qquad \cdot \qquad \cdot$$

$$u_p = \phi_p(x_1 \cdots x_m),$$

which are one-valued and continuous in a certain region $\mathfrak{A}$, *about the point*

$$a; \qquad x_1 = a_1 \cdots x_m = a_m;$$

and at this point, these functions have the values

$$u_1 = b_1 \cdots u_p = b_p.$$

Furthermore, the system S *admits no other set of* p *functions, continuous at* a *and taking on the initial values* b *at that point.*

We replace the system S by the equivalent system

$$\begin{aligned} d_{11}(u_1 - b_1) + \cdots + d_{1p}(u_p - b_p) &= d_{11}(u_1 - b_1) + \cdots \\ &\quad + d_{1p}(u_p - b_p) - F_1 = g_1 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad & \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ d_{p1}(u_1 - b_1) + \cdots + d_{pp}(u_p - b_p) &= d_{p1}(u_1 - b_1) + \cdots \\ &\quad + d_{pp}(u_p - b_p) - F_p = g_p. \end{aligned} \tag{2}$$

Since $\Delta \neq 0$, we can solve this system for the differences $u_\iota - b_\iota$, and get

$$\begin{aligned} u_1 - b_1 &= e_{11}g_1 + \cdots + e_{1p}g_p = f_1 \\ \cdot \quad \cdot \quad \cdot & \quad \cdot \quad \cdot \quad \cdot \\ u_p - b_p &= e_{p1}g_1 + \cdots + e_{pp}g_p = f_p. \end{aligned} \tag{3}$$

Obviously, the functions g, and hence the functions f, are continuous in D.

So are the derivatives $\frac{\partial f}{\partial u_\kappa}$. For

$$\frac{\partial f_\iota}{\partial u_\kappa} = e_{\iota 1}\frac{\partial g_1}{\partial u_\kappa} + \cdots + e_{\iota p}\frac{\partial g_p}{\partial u_\kappa}, \qquad \iota, \kappa = 1, 2 \cdots p. \quad (4$$

where

$$\frac{\partial g_r}{\partial u_\kappa} = d_{r\kappa} - \frac{\partial F_r}{\partial u_\kappa}. \quad (5$$

Since the g's vanish at Q, so do the f's. Since the derivatives 5) vanish at Q, so do also the derivatives 4).

Obviously, therefore, the numbers σ, τ, θ of lemma 440 exist, such that

$$\left|\frac{\partial f_\iota}{\partial u_\kappa}\right| < \frac{\theta}{p}, \qquad \text{in } D.$$

$$|f_\iota(x_1 \cdots x_m b_1 \cdots b_p)| < \tau(1-\theta), \qquad \text{in } \mathfrak{A}.$$

We can therefore apply this lemma to the system 3). Since this system and the given system S are equivalent, the theorem is proved.

442. 1. *Let*

$$f(x_1 \cdots x_m, u) = 0 \quad (1$$

admit a solution $u = b$, *at the point* $x = a$. *In* $D(a, b)$, *let* $f(x_1 \cdots x_m u)$ *have continuous first partial derivatives. Let* $f'_u \neq 0$ *in* D. *Then* 1) *defines a one-valued function* u, *in a certain domain* Δ, *of the point* a, *whose first partial derivatives in* Δ *are given by*

$$\frac{\partial u}{\partial x_\iota} = -\frac{f'_{x_\iota}}{f'_u}. \qquad \iota = 1, 2, \cdots m. \quad (2$$

For, let x be a point of Δ. Let x receive the increment Δx_ι, while the other coördinates of x remain constant. Let the corresponding increment of u be Δu. Then

$$f(x_1 \cdots x_\iota + \Delta x_\iota \cdots x_m u + \Delta u) - f(x_1 \cdots x_m u) = 0, \quad (3$$

by virtue of 1). Applying the Law of the Mean to 3), we have, setting $x'_\iota = x_\iota + \theta\Delta x_\iota$, $u' = u + \theta\Delta u$,

$$f'_{x_\iota}(x_1 \cdots x'_\iota \cdots x_m u')\Delta x_\iota + f'_u(x_1 \cdots x'_\iota \cdots x_m u')\Delta u = 0;$$

whence

$$\frac{\Delta u}{\Delta x_\iota} = -\frac{f'_{x_\iota}(x_1 \cdots x'_\iota \cdots x_m u')}{f'_u(x_1 \cdots x'_\iota \cdots x_m u')}.$$

Passing to the limit, we get 2).

2. $$df = \frac{\partial f}{\partial x_1} dx_1 + \cdots + \frac{\partial f}{\partial x_m} dx_m + \frac{\partial f}{\partial u} du = 0. \qquad \frac{\partial f}{\partial u} \neq 0. \qquad (4$$

For, by 2),

$$du = -\frac{\dfrac{\partial f}{\partial x_1}}{\dfrac{\partial f}{\partial u}} dx_1 - \cdots - \frac{\dfrac{\partial f}{\partial x_m}}{\dfrac{\partial f}{\partial u}} dx_m.$$

Multiplying by the common denominator, we have 4).

443. 1. *Let the system*

$$\begin{aligned} &F_1(x_1 \cdots x_m, u_1 \cdots u_p) = 0 \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \\ &F_p(x_1 \cdots x_m, u_1 \cdots u_p) = 0 \end{aligned} \qquad (1$$

admit a solution $u = b$ *at the point* $x = a$. *Let the functions* $F_1 \cdots F_p$ *have continuous first partial derivatives in* $D(a, b)$. *Let*

$$J = \begin{vmatrix} \dfrac{\partial F_1}{\partial u_1} \cdots \dfrac{\partial F_p}{\partial u_1} \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \dfrac{\partial F_1}{\partial u_p} \cdots \dfrac{\partial F_p}{\partial u_p} \end{vmatrix} \neq 0, \qquad \text{in } D.$$

Then 1) *defines a system* $u_1 \cdots u_p$ *of one-valued functions in a certain domain* Δ *of the point* a, *whose first partial derivatives* $\dfrac{\partial u_\iota}{\partial x_\kappa}$ *in* Δ *are given by the system of equations*

$$\begin{aligned} &\frac{\partial F_1}{\partial x_\kappa} + \frac{\partial F_1}{\partial u_1}\frac{\partial u_1}{\partial x_\kappa} + \cdots + \frac{\partial F_1}{\partial u_p}\frac{\partial u_p}{\partial x_\kappa} = 0 \\ &\cdot \quad \cdot \quad \cdot \\ &\frac{\partial F_p}{\partial x_\kappa} + \frac{\partial F_p}{\partial u_1}\frac{\partial u_1}{\partial x_\kappa} + \cdots + \frac{\partial F_p}{\partial u_p}\frac{\partial u_p}{\partial x_\kappa} = 0, \end{aligned} \qquad (2$$

with non-vanishing determinant J.

For, let $P=(x_1 \cdots x_m u_1 \cdots u_p)$ be a point of D. Let $\Delta u_1 \cdots \Delta u_p$ be the increments of $u_1 \cdots u_p$, corresponding to an increment Δx_κ of x_κ. Let $0<\theta_\iota<1$, and

$$Q_\iota=(x_1 \cdots x_\kappa+\theta_\iota \Delta x_\kappa \cdots x_m,\ u_1+\theta_\iota \Delta u_1 \cdots u_p+\theta_\iota \Delta u_p). \quad \iota=1, 2, \cdots p.$$

Let ϕ_ι be the value of $\dfrac{\partial F_\iota}{\partial x_\kappa}$, and $\psi_{\iota r}$ be the value of $\dfrac{\partial F_\iota}{\partial u_r}$ at Q_ι. Then, by the Law of the Mean, we have from 1),

$$\begin{aligned} \Delta F_1 &= \phi_1+\psi_{11} \frac{\Delta u_1}{\Delta x_\kappa}+\psi_{12} \frac{\Delta u_2}{\Delta x_\kappa}+\cdots+\psi_{1p} \frac{\Delta u_p}{\Delta x_\kappa}=0 \\ & \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \Delta F_p &= \phi_p+\psi_{p1} \frac{\Delta u_1}{\Delta x_\kappa}+\psi_{p2} \frac{\Delta u_2}{\Delta x_\kappa}+\cdots+\psi_{pp} \frac{\Delta u_p}{\Delta x_\kappa}=0. \end{aligned} \tag{3}$$

Thus,

$$\frac{\Delta u_\iota}{\Delta x_\kappa}=-\begin{vmatrix} \psi_{11} \cdots \phi_1 \cdots \psi_{1p} \\ \psi_{21} \cdots \phi_2 \cdots \psi_{2p} \\ \cdot \ \cdot \ \cdot \ \cdot \ \cdot \ \cdot \ \cdot \ \cdot \\ \psi_{p1} \cdots \phi_p \cdots \psi_{pp} \end{vmatrix} : \begin{vmatrix} \psi_{11} \cdots \psi_{1p} \\ \psi_{21} \cdots \psi_{2p} \\ \cdot \ \cdot \ \cdot \ \cdot \ \cdot \\ \psi_{p1} \cdots \psi_{pp} \end{vmatrix}.$$

Let $\Delta x_\kappa \doteq 0$; the limit of the right side exists, since the partial derivatives of the F's are continuous, and $J \neq 0$. Hence the derivatives $\dfrac{\partial u_\iota}{\partial x_\kappa}$ exist. Hence in the limit, the system 3) goes over into the system 2).

2. The determinant J is called the *Jacobian of the system* 1).

CHAPTER X

INDETERMINATE FORMS

Application of Taylor's Development in Finite Form

444. The object of the present chapter is to show how in certain cases we may determine the limit of expressions of the type

$$\frac{f(x)}{g(x)}, \quad f(x)\cdot g(x), \quad f(x)-g(x), \quad f(x)^{g(x)}$$

which, on replacing $f(x)$, $g(x)$ by their limits, assume the forms

$$\frac{0}{0}, \quad \frac{\infty}{\infty}, \quad 0\cdot\infty, \quad \infty-\infty, \quad 1^{\infty}, \quad 0^0, \quad \infty^0.$$

These are ordinarily called *indeterminate forms.*

445. Suppose by the aid of Taylor's development in finite form, or otherwise, we find that, in $R = RD(a)$,

$$f(x) = \alpha(x-a)^m + \phi(x)(x-a)^{m'}, \qquad m' > m.$$

$$g(x) = \beta(x-a)^n + \psi(x)(x-a)^{n'}, \qquad n' > n.$$

where ϕ, ψ are limited in R, and $\alpha, \beta \neq 0$.

Then

$$\frac{f(x)}{g(x)} = \frac{\alpha + (x-a)^{m'-m}\phi}{\beta + (x-a)^{n'-n}\psi}\cdot(x-a)^{m-n}. \qquad x \neq a.$$

Passing to the limit $x = a$, we have

$$R\lim_{x=a}\frac{f(x)}{g(x)} = \begin{cases} 0, & \text{if } m > n, \\ \alpha/\beta, & \text{if } m = n, \\ \sigma\cdot\infty, & \text{if } m < n. \qquad \sigma = \operatorname{sgn}\dfrac{\alpha}{\beta}. \end{cases}$$

Similar considerations apply to the left hand limit at a.

Example.

$$\lim_{x=0} \frac{e^x - e^{\sin x}}{x - \sin x} = +1.$$

For,

$$e^x = 1 + \frac{x}{1\,!} + \frac{x^2}{2\,!} + \frac{x^3}{3\,!} + x^4\phi_1(x).$$

$$e^{\sin x} = 1 + x + \tfrac{1}{2}x^2 + x^4\phi_2(x).$$

$$f(x) = e^x - e^{\sin x} = \tfrac{1}{6}x^3 + x^4\phi(x).$$

Similarly,

$$g(x) = x - \sin x = \tfrac{1}{6}x^3 + x^5\psi(x).$$

The functions ϕ, ψ are limited in $D(0)$.

Thus,

$$\frac{f(x)}{g(x)} = \frac{\frac{1}{6} + x\phi(x)}{\frac{1}{6} + x^2\psi(x)},$$

whose limit for $x = 0$ is 1.

446. To find the limit of

$$f(x) - g(x), \tag{1}$$

when f and g are infinite in the limit, we may sometimes find a development of $f(x)$, $g(x)$ in the form

$$\frac{\alpha_{-m}}{(x-a)^m} + \frac{\alpha_{-m+1}}{(x-a)^{m-1}} + \cdots + \alpha_0 + \alpha_1(x-a) + \cdots + (x-a)^n\phi(x),$$

valid in $D(a)$ or $RD(a)$, the function ϕ being limited here.

This method of finding the limit of 1) is best illustrated by an example.

$$\lim_{x=0}\left(\frac{1}{x} - \operatorname{cosec} x\right) = 0.$$

We have

$$\operatorname{cosec} x = \frac{1}{\sin x} = \frac{1}{x\{1 - x^2\phi(x)\}}, \qquad \phi(x) \doteq \tfrac{1}{6}.$$

$$\frac{1}{1 - x^2\phi(x)} = 1 + \frac{x^2\phi(x)}{1 - x^2\phi(x)} = 1 + x^2\psi(x), \qquad \psi(x) \doteq \tfrac{1}{6}.$$

Hence

$$\operatorname{cosec} x = \frac{1}{x}\{1 + x^2\psi(x)\}.$$

Therefore

$$\frac{1}{x} - \operatorname{cosec} x = -x\psi(x) \doteq 0.$$

447. When the independent variable $x \doteq +\infty$, we may set

$$x = \frac{1}{u},$$

which converts the limit into $R \lim\limits_{u=0}$, by 290.

Example. $$y = x(a^{\frac{1}{x}} - 1), \qquad a > 0.$$

$$\lim_{x=+\infty} y = \log a.$$

For, $$a^{\frac{1}{x}} = a^u = e^{u \log a} = 1 + u \log a + u^2 \phi(u),$$

where $$\phi(u) \doteq \tfrac{1}{2} \log^2 a.$$

Hence $$y = \log a + u\phi(u)$$

$$\doteq \log a.$$

448. When the preceding methods are not convenient, we may often apply with success one of the following theorems. These rest on

Cauchy's theorem. Let $f(x)$, $g(x)$ be continuous in $\mathfrak{A} = (a, b)$. Within $\mathfrak{A}$, let $f'(x)$ be finite or infinite and $g'(x)$ finite and $\neq 0$. Then

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}. \qquad a < c < b. \tag*{(1}$$

We note first that $g(b) \neq g(a)$. For, if $g(b) = g(a)$, we can apply Rolle's theorem to $g(x)$, which shows that $g'(x)$ must vanish within $\mathfrak{A}$, which is contrary to the hypothesis.

To prove 1), we introduce the auxiliary function

$$h(x) = f(x) - f(a) - \frac{f(b) - f(a)}{g(b) - g(a)} \{g(x) - g(a)\}.$$

Obviously, $h(x)$ is continuous in $\mathfrak{A}$. Also for points within $\mathfrak{A}$, for which $f'(x)$ is finite,

$$h'(x) = f'(x) - \frac{f(b) - f(a)}{g(b) - g(a)} g'(x);$$

while for the other points within $\mathfrak{A}$, $h'(x)$ is definitely infinite. Finally, we observe that $h(a) = h(b) = 0$. We can thus apply Rolle's theorem to $h(x)$, which gives 1) at once.

449. 1. *Let* $f(x), f'(x) \cdots f^{(n-1)}(x),\ g(x), g'(x) \cdots g^{(n-1)}(x)$ *be continuous in* $\mathfrak{A} = (a, a+\delta)$, *and vanish for* $x = a$. *Let* $f^{(n)}(x)$ *be finite or infinite within* $\mathfrak{A}$. *Let* $g^{(n)}(x)$ *be finite and* $\neq 0$ *within* $\mathfrak{A}$. *Let* $g'(x), g''(x) \cdots g^{(n-1)}(x) \neq 0$ *within* $\mathfrak{A}$. *Then*

$$\frac{f(a+h)}{g(a+h)} = \frac{f^{(n)}(c)}{g^{(n)}(c)}. \qquad a < c < a+h. \tag{1}$$

For, by 448,

$$\frac{f(a+h)}{g(a+h)} = \frac{f'(c_1)}{g'(c_1)}, \qquad a < c_1 < a+h;$$

$$\frac{f'(c_1)}{g'(c_1)} = \frac{f''(c_2)}{g''(c_2)}, \qquad a < c_2 < c_1, \quad \text{etc.}$$

2. We note that the denominator $g(a+h)$ in 1) is $\neq 0$. For otherwise, $g'(x)$ would vanish somewhere within $\mathfrak{A}$.

The Form $\frac{0}{0}$

450. 1. *Let* $f(x), g(x)$ *be continuous in* $R = RD(a)$, *and vanish at* a. *Let* $g'(x)$ *be finite, and* $\neq 0$ *within* R. *Let* $f'(x)$ *be finite or infinite within* R. *Let*

$$R \lim_{x=a} \frac{f'(x)}{g'(x)} = \lambda, \qquad \textit{finite or infinite.} \tag{1}$$

where x *runs over only those values for which* $f'(x)$ *is finite.*

Then

$$R \lim_{x=a} \frac{f(x)}{g(x)} = \lambda. \tag{2}$$

For, by 449,

$$\frac{f(x)}{g(x)} = \frac{f'(\xi)}{g'(\xi)}. \qquad a < \xi < x.$$

The limit of the right side, as $x \doteq a$, is λ.

Hence the limit* of the left side is λ, for $x = a$.

---

* The reader should bear in mind that a limit is a general limit, unless the contrary is stated. Thus in 2), x runs over all values within R as it $\doteq a$; while in 1) it ranges only over a specified part of R.

2. *Let $f(x)$, $g(x)$ vanish at $x=a$, while $g(x)\neq 0$ within $RD(a)$. Let $f'(a)$ exist, finite or infinite. Let $g'(a)$ exist and be $\neq 0$. Then*

$$R\lim_{x=a}\frac{f(x)}{g(x)}=\frac{f'(a)}{g'(a)}.$$

For,

$$\frac{f(x)}{g(x)}=\frac{\dfrac{f(x)-f(a)}{x-a}}{\dfrac{g(x)-g(a)}{x-a}}\doteq\frac{f'(a)}{g'(a)}.$$

3. We can generalize 2 as follows:

Let $f(x)$, $g(x)$ and their first $n-2$ derivatives be continuous in $R=RD(a)$. Within R, let $f^{(n-1)}(x)$ be finite or infinite, $g^{(n-1)}(x)$ finite, and $g', g'' \cdots g^{(n-1)}\neq 0$. Let f, g and their first $n-1$ derivatives vanish at a. Let $f^{(n)}(a)$ be finite or infinite, while $g^{n}(a)$ is finite and $\neq 0$. Then

$$\lim_{x=a}\frac{f(x)}{g(x)}=\frac{f^{(n)}(a)}{g^{(n)}(a)}. \tag{2}$$

For, by 449,

$$\frac{f(x)}{g(x)}=\frac{f^{(n-1)}(c)}{g^{(n-1)}(c)}=\frac{\dfrac{f^{(n-1)}(c)-f^{(n-1)}(a)}{c-a}}{\dfrac{g^{(n-1)}(c)-g^{(n-1)}(a)}{c-a}}.$$

But as $x\doteq a$, so does $c\doteq a$. Hence, passing to the limit $x=a$, we get 2).

Example. Let $f(x)=x^2$, for rational x.

$= 0$, for irrational x.

Let $g(x)=\sin x$.

Here $f'(x)$ does not exist except at $x=0$, where it $=0$. Hence, by 2,

$$R\lim_{x=0}\frac{f(x)}{g(x)}=\frac{f'(0)}{g'(0)}=\frac{0}{1}=0,$$

a result which is obvious from other considerations.

4. In 1, we assume the existence of $\lambda=R\lim\dfrac{f'(x)}{g'(x)}$, and then show that

$$R\lim\frac{f(x)}{g(x)}=\lambda. \tag{1}$$

That the limit on the left side of 1) can exist when λ does not is shown by the example in 3. It is also illustrated by the following:

Let $$f(x) = x^2 \sin\frac{1}{x}, \text{ for } x \neq 0.$$
$$= 0, \qquad \text{for } x = 0.$$

Let $$g(x) = x.$$

Then, for $x \neq 0$,
$$f'(x) = 2x \sin\frac{1}{x} - \cos\frac{1}{x};$$

while
$$g'(x) = 1.$$

Hence
$$\lambda = R \lim_{x=0} \frac{f'(x)}{g'(x)}$$

does not exist. On the other hand,
$$\lim_{x=0} \frac{f(x)}{g(x)} = 0.$$

We observe that this result also follows from 2.

451. Suppose:

1°. *$f(x)$, $g(x)$ are continuous in $D(+\infty)$;*

2°. *$f'(x)$ is finite or infinite in D;*

3°. *$g'(x)$ is finite and $\neq 0$ in D;*

4°. $f(+\infty) = g(+\infty) = 0.$

Let $$\lim_{x=+\infty} \frac{f'(x)}{g'(x)} = \lambda, \qquad \lambda \text{ finite or infinite}$$

where x runs over only those values for which $f'(x)$ is finite.

Then * $$\lim_{x=+\infty} \frac{f(x)}{g(x)} = \lambda.$$

We set $$x = \frac{1}{u}.$$

Then D goes over into $R = RD^*(0)$.

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* Cf. footnote, page 301.

Let

$$f(x) = f\left(\frac{1}{u}\right) = \phi(u);$$

$$g(x) = g\left(\frac{1}{u}\right) = \psi(u).$$

The functions ϕ, ψ not being defined for $u = 0$, we set

$$\phi(0) = \psi(0) = 0. \tag{1}$$

Since f, g are continuous in D, ϕ, ψ are continuous in R, by virtue of 1) and 4°.

For points of D at which $f'(x)$ is finite,

$$\phi'(u) = \frac{df}{dx}\frac{dx}{du} = -f'(x) \cdot x^2.$$

Hence at the corresponding points u in R, $\phi'(u)$ is finite.

From the relation

$$\frac{\Delta\phi}{\Delta u} = \frac{\Delta f}{\Delta x} \cdot \frac{\Delta x}{\Delta u},$$

we see that when $f'(x)$ is definitely infinite in D, $\phi'(u)$ is also infinite at the corresponding u point in R.

Thus $\phi'(u)$ is finite or infinite in R, while $\psi'(u)$ is finite and $\neq 0$ there.

Then by 450, 1, if

$$R\lim_{u=0} \frac{\phi'(u)}{\psi'(u)} = \lambda, \qquad \lambda \textit{ finite or infinite.}$$

u running over only those points for which $\phi'(u)$ is finite,

$$R\lim_{u=0} \frac{\phi(u)}{\psi(u)} = \lambda.$$

But

$$R\lim_{u=0} \frac{\phi(u)}{\psi(u)} = \lim_{x=+\infty} \frac{f(x)}{g(x)}. \tag{1}$$

Also

$$\lambda = R\lim_{u=0} \frac{\phi'(u)}{\psi'(u)} = \lim_{x=+\infty} \frac{x^2 f'(x)}{x^2 g'(x)} = \lim_{x=+\infty} \frac{f'(x)}{g'(x)}. \tag{2}$$

Hence 1), 2) give the theorem.

The Form $\frac{\infty}{\infty}$

452. *Let* $f(a+0)$, $g(a+0)$ *be infinite.*
In $R = RD(a)$ *suppose that*

1°. $f(x)$, $g(x)$ *are continuous;*
2°. $f'(x)$ *is finite or infinite;*
3°. $g'(x)$ *is finite and* $\neq 0$.

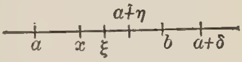
Let

$$R \lim_{x=a} \frac{f'(x)}{g'(x)} = \lambda, \qquad \lambda \textit{ finite or infinite.}$$

x ranging over only those values for which $f'(x)$ is finite.
Then *

$$R \lim_{x=a} \frac{f(x)}{g(x)} = \lambda.$$

Let $a < x < b < a + \delta$. Then, by 448,



$$\frac{f(x)-f(b)}{g(x)-g(b)} = \frac{f'(\xi)}{g'(\xi)}, \qquad x < \xi < b.$$

Thus

$$f(x) = f(b) + \frac{f'(\xi)}{g'(\xi)} \{g(x) - g(b)\},$$

whence

$$\frac{f(x)}{g(x)} = \frac{f(b)}{g(x)} + \frac{f'(\xi)}{g'(\xi)} \left\{ 1 - \frac{g(b)}{g(x)} \right\}. \qquad (1$$

Here b is any fixed point in R.

There are two cases according as λ is finite or infinite.

Suppose λ is finite. Let $\sigma > 0$ be small at pleasure; we can take δ so small that

$$\frac{f'(\xi)}{g'(\xi)} = \lambda + \sigma'. \qquad |\sigma'| < \sigma.$$

Let $\tau > 0$ be small at pleasure. We can choose $a + \eta < b$, such that

$$\tau' = \frac{f(b)}{g(x)}, \qquad \tau'' = \frac{g(b)}{g(x)}$$

are numerically $< \tau$, if $x < a + \eta < b$.

---

* Cf. footnote, page 301.

Then for all x in $(a^*, a+\eta)$, we have by 1),

$$\frac{f(x)}{g(x)} = \tau' + (\lambda + \sigma')(1 - \tau''). \tag{2}$$

Thus

$$\left|\frac{f(x)}{g(x)} - \lambda\right| < \tau(1 + |\lambda|) + \sigma(1 + \tau) < \epsilon,$$

if σ and τ are taken sufficiently small.

Suppose λ *is infinite, say* $\lambda = +\infty$. Let $M > 0$ be large at pleasure. We can choose δ so small that

$$\frac{f'(\xi)}{g'(\xi)} = M(1 + \mu), \qquad \mu > 0.$$

Choosing η as before, we have for every x in $(a^*, a+\eta)$,

$$\frac{f(x)}{g(x)} = \tau' + M(1 + \mu)(1 - \tau'').$$

If we suppose $\tau < \frac{1}{2}$, and M sufficiently large,

$$\frac{f(x)}{g(x)} > M(1 - \tau) - 1 > G, \qquad \text{in } D_\eta^*,$$

where G is as large as we please.

453. *Let* $f(+\infty)$, $g(+\infty)$ *be infinite.*

In $D(+\infty)$, *let*

1°. $f(x)$, $g(x)$ *be continuous;*
2°. $f'(x)$ *be finite or infinite;*
3°. $g'(x)$ *be finite and* $\neq 0$.

Let

$$\lim_{x=+\infty} \frac{f'(x)}{g'(x)} = \lambda, \qquad \lambda \text{ finite or infinite.}$$

Then

$$\lim_{x=+\infty} \frac{f(x)}{g(x)} = \lambda.$$

We deduce this theorem from 452 in the same way as 451 was derived from 450.

The Forms $0\cdot\infty$, $\infty-\infty$, 0^0, 1^∞, ∞^0

454. 1. Let $f(x)\doteq 0$, $g(x)\doteq\pm\infty$. Then $f(x)g(x)$ is of the form $0\cdot\infty$.

Setting $fg=\dfrac{f}{\frac{1}{g}}$, this form is reduced to $\dfrac{0}{0}$.

2. Let $f(x)\doteq\pm\infty$, $g(x)\doteq\pm\infty$, the infinities having same signs. Then $f(x)-g(x)$ is of the form $\infty-\infty$.

Setting

$$f-g=\frac{\frac{1}{g}-\frac{1}{f}}{\frac{1}{fg}};$$

this form is reduced to $\dfrac{0}{0}$.

3. Let $f\doteq 0$, $g\doteq 0$. Then $[f(x)]^{g(x)}$ is of the form 0^0.

Let
$$y=f^g,\qquad f(x)>0.$$
Then
$$\log y=g\log f=\frac{g}{\frac{1}{\log f}}$$
is of the form $\dfrac{0}{0}$.

If
$$\log y\doteq\lambda,$$
then
$$\lim y=\lim\,[f(x)]^{g(x)}=e^{\lambda}.$$

The other forms 1^∞, ∞^0 are treated in a similar manner.

EXAMPLES

1.
$$x^{\mu}\log(1-\cos x),\qquad \mu,\ x>0. \tag{1}$$
has the form $0\cdot\infty$ for $x=0$. We may write it

$$\frac{\log(1-\cos x)}{x^{-\mu}},$$

which has the form $\frac{\infty}{\infty}$. The conditions of 452 being satisfied, we differentiate numerator and denominator, getting as new quotient

$$-\frac{\frac{\sin x}{1-\cos x}}{\mu x^{-(1+\mu)}}=-\frac{1}{\mu}\,\frac{x^{1+\mu}}{\tan\frac{x}{2}}.$$

This has the form $\frac{0}{0}$, for $x = 0$. Applying 450, 1, we get, differentiating once more,

$$-2\frac{\mu+1}{\mu}\frac{x^\mu}{\sec^2\frac{x}{2}},$$

whose limit for $x = 0$, is 0.

Hence,

$$R\lim_{x=0} x^\mu \log(1-\cos x) = 0.$$

2. $$R\lim_{x=0} x^a |\log x|^\mu = 0. \qquad a,\ \mu > 0. \tag{2}$$

For,

$$x^a|\log x|^\mu = \frac{|\log x|^\mu}{x^{-a}} = \frac{f(x)}{g(x)}.$$

We apply 452.

$$\frac{f'(x)}{g'(x)} = \frac{\mu}{a}\frac{|\log x|^{\mu-1}}{x^{-a}}.$$

If $\mu \leq 1$, this expression $\doteq 0$. If $\mu > 1$, we differentiate again, etc.

3. At first sight one might think that

$$\lim_{n=\infty}\frac{f(n+1)}{f(n)} = 1, \tag{3}$$

since $\frac{n+1}{n} \doteq 1$. This is, however, not true in general.

For example, let $f(x) = e^x$.

Then

$$\frac{f(n+1)}{f(n)} = \frac{e^{n+1}}{e^n} = e.$$

Hence the limit 3) is here e and not 1.

Again, let

$$f(x) = e^{e^x}.$$

Then

$$\frac{f(n+1)}{f(n)} = \frac{e^{e^{n+1}}}{e^{e^n}} = e^{e^n(e-1)},$$

which $\doteq +\infty$.

Criticisms

455. 1. The treatment of indeterminate forms in many textbooks is deplorable.

We consider some of the objectionable points in detail.

When $f(x)$, $g(x)$ vanish at $x = a$, the function

$$\phi(x) = \frac{f(x)}{g(x)} \tag{1}$$

is not defined at a.

Some authors admit division by 0.

From this standpoint the value of ϕ at a is hidden because ϕ takes on the indeterminate form $\frac{0}{0}$. The *true value*, as such authors say, may often be found by a simple transformation, or by the method of limits.

For example, if

$$f(x) = x^2 - a^2, \quad g(x) = x - a,$$

the *true value* of ϕ may be found by removing the common factor $x - a$ in

$$\frac{x^2 - a^2}{x - a} = (x + a)\frac{x - a}{x - a}.$$

Thus

$$\phi(a) = 2a.$$

As already remarked, division by 0 is ruled out in modern analysis.

First, because it is nowhere necessary; and secondly, because of the difficulties and ambiguities it gives rise to.

The expression 1) has then no value assigned to it for $x = a$. We may therefore, if we choose, agree that in all such cases ϕ shall have the value

$$\lim_{x=a} \frac{f(x)}{g(x)}$$

when this is finite. Some authors do this; in this case ϕ has a *true value* at a. However, we shall make no such convention in this work.

2. In this connection let us give an example of the so-called paradoxes which arise from division by 0.

Let $x = 1$; then

$$x^2 - 1 = x - 1.$$

Dividing both sides by $x - 1$, we get

$$x + 1 = 1,$$

which gives, since $x = 1$,

$$2 = 1.$$

It is easy to see where the trouble arises.

When $c \neq 0$, we can always conclude from

$$ac = bc \tag{2}$$

that

$$a = b. \tag{3}$$

If, however, $c = 0$, we cannot always conclude 3) from 2). In fact, take $a \neq b$, if $c = 0$, we still have

$$ac = bc.$$

456. 1. To find $\lim \phi(x)$, some writers proceed thus. From

$$\phi(a+h) = \frac{f(a+h)}{g(a+h)} = \frac{\dfrac{f(a+h)-f(a)}{h}}{\dfrac{g(a+h)-g(a)}{h}},$$

they conclude that

$$\lim \phi = \frac{f'(a)}{g'(a)}. \tag{1}$$

This is correct if $f'(a)$, $g'(a)$ exist, and the latter is $\neq 0$. If both are 0, they say

$$\frac{f'(a)}{g'(a)} \tag{2}$$

is still indeterminate. Applying the preceding reasoning to 2), it follows that its *true* value is that of

$$\frac{f''(a)}{g''(a)}.$$

This last step would be permissible, provided the first step showed that

$$\lim \phi = \lim \frac{f'(x)}{g'(x)}. \tag{3}$$

But it does not; it shows only that 1) is true, and even here we must assume that $g'(a) \neq 0$.

In order to take this second step correctly, we have proved Cauchy's theorem, 448. Cf. 449, 450.

2. In this connection we note that we cannot always say that

$$\frac{f(x)}{g(x)} \text{ and } \frac{f'(x)}{g'(x)}$$

have the same limit for $x=a$, when $f(a)=g(a)=0$.

For example, let

$$f(x)=x^2\sin\frac{1}{x},\ g(x)=x. \qquad a=0.$$

Then

$$\lim_{x=0}\frac{f(x)}{g(x)}=0,$$

while

$$\frac{f'(x)}{g'(x)}=2x\sin\frac{1}{x}-\cos\frac{1}{x} \qquad x>0.$$

has no limit for $x=0$. Cf. 450.

457. Some writers, using the relation of Cauchy,

$$\phi=\frac{f(x)}{g(x)}=\frac{f'(c)}{g'(c)}, \qquad a<c<x.$$

conclude now that

$$\lim\phi=\frac{f'(a)}{g'(a)}.$$

This is true if $f'(x)$, $g'(x)$ are continuous at a, and $g'(a)\neq 0$.

458. Some writers, in order to evaluate $\lim\phi$, develop $f(x)$, $g(x)$ into *infinite power series*. The *possibility* of such a development is established only for a few simple cases in many text-books. For example, such books do not show that

$$\sec x, \quad \tan x, \quad e^{\sin x}$$

can be developed into power series; yet they give examples of indeterminate forms involving these functions.

There is, however, no necessity of using infinite series; all that is needed for such cases is Taylor's development in *finite* form. See 445, 446.

459 To evaluate the form $\frac{\infty}{\infty}$, some writers proceed thus:

$$\phi(x) = \frac{f(x)}{g(x)} = \frac{\dfrac{1}{g(x)}}{\dfrac{1}{f(x)}}.$$

Hence

$$\lim \phi(x) = \lim \frac{\dfrac{g'(x)}{g^2(x)}}{\dfrac{f'(x)}{f^2(x)}} = \lim \phi^2(x) \frac{g'(x)}{f'(x)}.$$

Dividing by $\lim \phi(x)$, they get

$$1 = \lim \phi(x) \frac{g'(x)}{f'(x)}.$$

Hence

$$\lim \phi(x) = \lim \frac{f'(x)}{g'(x)}.$$

This method assumes the existence of $\lim \phi(x)$; that is, the existence of the very thing we are seeking is put in question. Suppose by this method we find that

$$\lim \frac{f'(x)}{g'(x)} = 1$$

for example; what right have we to say that therefore

$$\lim \frac{f(x)}{g(x)} = 1?$$

None whatever, until by some subsidiary investigation, the existence of $\lim \phi$ is established. See 377, 3.

Scale of Infinitesimals and Infinities

460. Consider the functions

$$f(x) = \log^{\alpha} x, \quad g(x) = x^{\beta}, \qquad \alpha, \beta > 0.$$

Both increase indefinitely as $x \doteq +\infty$. We may ask which increases faster.

The quotient

$$Q = \frac{f(x)}{g(x)}$$

is of the form $\frac{\infty}{\infty}$. The conditions of 453 being satisfied, we consider

$$Q_1 = \frac{f'(x)}{g'(x)} = \frac{\alpha}{\beta} \frac{\log^{\alpha-1} x}{x^\beta}.$$

If $0 < \alpha \leq 1$, $Q_1 \doteq 0$; hence $Q \doteq 0$.

If $\alpha > 1$, we consider

$$Q_2 = \frac{f''(x)}{g''(x)} = \frac{\alpha \cdot \overline{\alpha - 1}}{\beta^2} \frac{\log^{\alpha-2} x}{x^\beta}.$$

Thus if $0 < \alpha \leq 2$, $Q_2 \doteq 0$; hence $Q \doteq 0$.

If $\alpha > 2$, we may continue this process. As the exponent α is diminished by unity each time, $\log x$ must have finally a negative or zero exponent. Thus in every case $Q \doteq 0$.

461. 1. Let $f(x)$, $g(x)$ become infinite for $x \doteq a$, a being finite or infinite. If

$$\lim \frac{f(x)}{g(x)} \text{ is finite and } \neq 0,$$

we say f and g are of the *same order infinite*. If

$$\lim \frac{f(x)}{g(x)} = 0,$$

we say f is of *lower order infinite than g*. If

$$\lim \frac{f(x)}{g(x)} \text{ is infinite,}$$

we say f is of *higher order infinite than g*.

These three cases are denoted respectively by

$$f(x) \sim g(x), \quad f(x) < g(x), \quad f(x) > g(x).$$

We may also say more briefly, that $f(x)$ *is infinitarily equal, less than, greater than* $g(x)$.

2. Similar definitions hold when

$$f(x) \doteq 0, \quad g(x) \doteq 0.$$

If, for example,

$$\lim \frac{f(x)}{g(x)} = 0,$$

we say $f(x)$ is *infinitely small relative to* $g(x)$, or an *infinitesimal of higher order than* $g(x)$. We may also say $f(x)$ *is infinitarily smaller than* $g(x)$. In symbols,

$$f(x) < g(x).$$

3. Turning to the result of 460, we have:

$$\log^a x < x^\beta \qquad \alpha, \beta > 0, \ x \doteq +\infty.$$

however large α *is and however small* β.

462. Let us consider now functions of the type

$$l_m x, \quad l_1 x l_2 x l_3 x, \quad \cdots$$

For $x > 1$, we have

$$0 < \log x < x.$$

For sufficiently large x,

$$l_1 x, \quad l_2 x, \quad l_3 x \cdots l_m x$$

are > 0. We have, then,

$$x > l_1 x > l_2 x \cdots > l_m x.$$

The values of these *iterated* logarithms decrease very rapidly. For example, let

$$x = 1{,}000{,}000{,}000 = 10^9.$$

Then

$$l_1 x = 20.723, \ l_2 x = 3.031, \ l_3 x = 1.108, \ l_4 x = 0.103 \ \cdots$$

$$l_5 x = \text{a negative number.}$$

Hence $l_6 x$ does not exist.

463. 1. *When* $x \doteq +\infty$, *we have, if* $\alpha, \alpha_1, \alpha_2 \cdots > 0$,

$$x^\alpha > l_1^{\alpha_1} x > l_2^{\alpha_2} x > l_3^{\alpha_3} x \cdots \qquad (S$$

The sequence S may be called the *logarithmic scale.*

To prove S, let

$$u = l_{m-1}x.$$

Then $u \doteq +\infty$ with x. We have

$$\lim_{x=+\infty} \frac{l_m{}^{a_m}x}{l_{m-1}{}^{a_{m-1}}x} = \lim_{u=+\infty} \frac{\log^{a_m} u}{u^{a_{m-1}}} = 0, \qquad m = 1, 2, \cdots$$

by 461, 3.

2. *If* $\alpha_1, \alpha_2, \cdots > 1$, $x \doteq +\infty$;

$$l_1{}^{\alpha_1}x > l_1 x l_2{}^{\alpha_2}x > l_1 x l_2 x l_3{}^{\alpha_3}x > \cdots$$

This follows at once from 1.

464. *Let* $x \doteq +\infty$, *while* $\alpha, \alpha_1, \alpha_2, \cdots > 0$. *Then*

$$x^\alpha < (e^x)^{\alpha_1} < \left(e^{e^x}\right)^{\alpha_2} < \left(e^{e^{e^x}}\right)^{\alpha_3} < \cdots \tag{T}$$

The sequence T may be called the *exponential scale*.

Let

$$f(x) = x^\alpha, \quad g(x) = e^{\alpha_1 x}, \quad Q = \frac{f(x)}{g(x)}.$$

We apply 453.

$$\frac{f'(x)}{g'(x)} = \frac{\alpha x^{\alpha-1}}{\alpha_1 e^{\alpha_1 x}}.$$

If now $0 < \alpha \leq 1$, $Q \doteq 0$.

If $\alpha > 1$,

$$\frac{f''(x)}{g''(x)} = \frac{\alpha(\alpha-1)x^{\alpha-2}}{\alpha_1{}^2 e^{\alpha_1 x}}.$$

If $1 < \alpha \leq 2$, this shows that $Q \doteq 0$, and so on.

Hence

$$x^\alpha < (e^x)^{\alpha_1}.$$

To show

$$(e^x)^{\alpha_1} < \left(e^{e^x}\right)^{\alpha_2}, \tag{1}$$

let us set

$$u = e^x.$$

Then

$$\lim_{x=+\infty} \frac{(e^x)^{\alpha_1}}{\left(e^{e^x}\right)^{\alpha_2}} = \lim_{u=+\infty} \frac{u^{\alpha_1}}{(e^u)^{\alpha_2}} = 0,$$

as just shown. This proves 1). The rest of the theorem follows now in the same way.

Order of Infinitesimals and Infinities

465. 1. Let $x \doteq 0$; then x is an infinitesimal; $x^2, x^3 \cdots$ are also infinitesimals.

Taking x as a standard, we may say x^n is an infinitesimal of *order* n, n being a positive integer, and, in general, if $x > 0$, x^μ is an infinitesimal of *order* μ, where μ is *any* positive number.

Then, if

$$R \lim_{x=0} \frac{f(x)}{x^\mu}$$

is finite and $\neq 0$, we say that $f(x)$ *is an infinitesimal of order* μ.

Not every infinitesimal, however, has an order.

For example, by 464, there is no number μ, such that

$$R \lim_{x=0} \frac{e^{-\frac{1}{x}}}{x^\mu}$$

is not 0. Hence $e^{-\frac{1}{x}}$ has no order.

2. On the other hand, an infinitesimal $f(x)$ may not have an order μ, because

$$R \lim_{x=0} \frac{f(x)}{x^\mu}$$

either does not exist, or when it does it is infinite or zero.

Thus

$$R \lim_{x=0} \frac{x \sin \frac{1}{x}}{x^\mu}$$

does not exist. Hence

$$x \sin \frac{1}{x}$$

is an infinitesimal without an order.

3. Obviously, similar remarks hold for infinities.

CHAPTER XI

MAXIMA AND MINIMA

ONE VARIABLE

Definition. Geometric Orientation

466. Let $f(x)$ be defined in $\mathfrak{A} = (a, b)$. Let c be an inner point of $\mathfrak{A}$. If

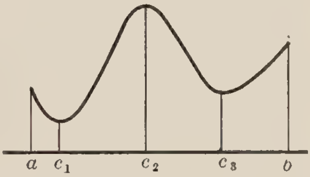
$$\Delta f = f(x) - f(c) > 0 \text{ in } D^*(c), \tag{1}$$

f has a *minimum* at c. If

$$\Delta f = f(x) - f(c) < 0 \text{ in } D^*(c), \tag{2}$$

f has a *maximum* at c.

In words, we may say: f has a maximum at c when $f(c)$ is greater than any other value of f in the domain of c; it has a minimum at c when $f(c)$ is less than any other value of f in the domain of c. According to this definition, $f(x)$, whose graph is given in the figure, has a maximum at c_2, and a minimum at c_1, c_3.



The reader should not confuse the terms $f(x)$ has a maximum or a minimum at a point c, with the terms

$$\operatorname{Min} f(x), \ \operatorname{Max} f(x), \text{ in } \mathfrak{A}.$$

A function may have an infinite number of *extremes*, that is, maxima or minima, in $\mathfrak{A}$.

Example.

$$f(x) = x^2\left(1 + \sin^2 \frac{1}{x}\right), \qquad \text{for } x \neq 0,$$
$$= 0. \qquad \text{for } x = 0.$$

This function oscillates between the two parabolas

$$y = x^2, \qquad y = 2x^2.$$

At the origin, f has a minimum; and in any vicinity of the origin, f has an infinite number of maxima and minima.

467. We consider now how the points at which $f(x)$ has an extreme may be determined. Consulting the Fig. in 466, the reader will observe that at the points of extreme the tangent is parallel to the axis of x; that is, at these points $f'(x) = 0$.

However, $f(x)$ does not need to have an extreme at all the points at which $f'(x) = 0$.

For example,

$$y = x^3.$$

This is an increasing function whose derivative vanishes at $x = 0$. In fact, at this point the graph has a point of inflection with a tangent parallel to the x axis. See Fig. 1.

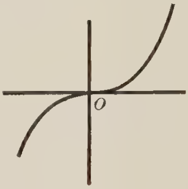


FIG. 1.

On the other hand, not all the points of extreme are given by the roots of $f'(x) = 0$.

Example.

$$y = x^{\frac{2}{3}}.$$

This function has a minimum at the origin O, which is a cuspidal point, with vertical tangent. See Fig. 2.

At this point, y has no differential coefficient, since

$$Rf'(0) = +\infty, \quad Lf'(0) = -\infty.$$

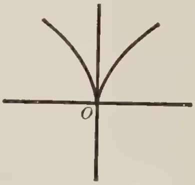


FIG. 2.

Criteria for an Extreme

468. 1. *In $D(a)$, let $f^{(n)}(x)$ be continuous and $f^{(n)}(a) \neq 0$. Let*

$$f'(a) = f''(a) = \cdots = f^{(n-1)}(a) = 0.$$

Then f has no extreme at a, if n is odd. If n is even, it has a minimum, if $f^{(n)}(a) > 0$; a maximum, if $f^{(n)}(a) < 0$.

For, under these conditions, we have

$$\Delta f = f(a+h) - f(a) = \frac{h^n}{n!} f^{(n)}(a+\theta h).$$

Since $f^{(n)}(x)$ is continuous at a,

$$\operatorname{sgn} f^{(n)}(a+\theta h) = \operatorname{sgn} f^{(n)}(a) = \sigma.$$

If n is odd,

$$\operatorname{sgn} \Delta f = \sigma \operatorname{sgn} h.$$

As h can take on positive and negative values, Δf does not preserve one sign in $D^*(a)$. Hence, f has no extreme at a. If n is even,

$$\operatorname{sgn} \Delta f = \sigma.$$

Thus in D^*,

$$\Delta f > 0, \text{ if } f^{(n)}(a) > 0,$$

$$< 0, \text{ if } f^{(n)}(a) < 0.$$

2. *In* $\mathfrak{A} = (a, b)$, *let* $f'(x)$ *exist, finite or infinite. The points within* $\mathfrak{A}$ *at which* $f(x)$ *has an extreme, lie among the zeros of* $f'(x)$.

For, suppose $f(x)$ has a maximum at c. Then for $h > 0$,

$$f(c+h) - f(c) < 0, \; f(c-h) - f(c) < 0.$$

Thus,

$$R = \frac{f(c+h) - f(c)}{h} < 0, \; L = \frac{f(c-h) - f(c)}{-h} > 0. \tag{1}$$

But when $h \doteq 0$,

$$\lim R = \lim L = f'(c). \tag{2}$$

On the other hand, 1) shows that

$$\lim R \leqq 0, \; \lim L \geqq 0;$$

which with 2) shows that $f'(c) = 0$.

3. The reasoning in 2 also shows:

If $f(x)$ *has an extreme at* $x = a$, *then* $f'(a) = 0$, *if* $f'(a)$ *exists*

469. *Let* $f(x)$ *be continuous in* $D(a)$. *Let* $f'(x)$ *be finite in* $D^*(a)$. *Let*

$$Rf'(a)=\sigma\infty,\ Lf'(a)=-\sigma\infty,\quad \sigma=\pm 1.$$

Then f *has a minimum at* a, *if* $\sigma=+1$; *and a maximum, if* $\sigma=-1$.

To fix the ideas, let $\sigma=+1$.

Then

$$R\lim_{h=0}\frac{f(a+h)-f(a)}{h}=+\infty,$$

$$R\lim_{h=0}\frac{f(a-h)-f(a)}{-h}=-\infty.$$

Thus there exists a $\delta>0$, such that

$$f(a+h)-f(a)>0,\qquad |h|<\delta.$$

in $D_\delta^*(a)$. Hence f has a minimum at a.

470. *Let* $f(x)$ *be continuous in* $D(a)$. *In* $D^*(a)$, *let* $f'(x)$ *be finite or infinite, and never vanish throughout any interval of it. In* $RD^*(a)$, *let* $f'(x)$ *be positive when not zero; in* $LD^*(a)$, *let* $f'(x)$ *be negative when not zero. Then* $f(x)$ *has a minimum at* a. *If these signs are reversed,* f *has a maximum.*

For, using 403, we see that $f(x)$ is an increasing function in $RD(a)$, and a decreasing function in $LD(a)$. Hence f has a minimum at a.

EXAMPLES

471.

$$f(x)=a+x^{\frac{2}{3}}.$$

In 388, we saw

$$Rf'(0)=+\infty,\qquad Lf'(0)=-\infty.$$

Hence, by 469, f has a minimum at 0, a result which may be seen directly.

472.

$$f(x)=\frac{e^{\frac{1}{x}}-1}{e^{\frac{1}{x}}+1},\ \text{for } x\neq 0,$$

$$=0,\qquad \text{for } x=0.$$

This function was considered in 366.

Applying 470, we see that f has a minimum at the origin, a result that may be seen directly.

473. Let
$$f(x) = e^{-\frac{1}{x^2}}, \text{ for } x \neq 0,$$
$$= 0, \quad \text{ for } x = 0.$$

This function is Cauchy's function. We see directly that it has a minimum at the origin. The same result is obtained by 470.

Criticism

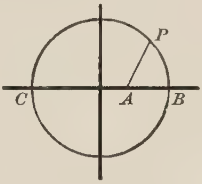
474. Some writers confound the terms the function has a maximum or minimum at a point, with the terms maximum or minimum of a function in an interval. These two terms may or may not mean the same thing.

For example, let
$$f(x) = \sin x, \qquad \mathfrak{A} = (0, 2\pi).$$

Then f has a maximum at $\frac{\pi}{2}$, and a minimum at $\frac{3\pi}{2}$. These are also the maximum and minimum of f in $\mathfrak{A}$. On the other hand, if we take $\mathfrak{B} = \left(0, \frac{\pi}{2}\right)$ as one interval, f has neither a maximum nor a minimum at any point in $\mathfrak{B}$; yet its maximum in $\mathfrak{B}$ is 1, and its minimum in $\mathfrak{B}$ is 0.

475. The following example also illustrates this point.

Find the greatest and least distance δ between a fixed point A within a circle and any point P on the circle.



Let the circle be
$$x^2 + y^2 = r^2,$$

and the coördinates of A be $a, 0$; $a > 0$.

Then
$$\delta = \sqrt{(x-a)^2 + y^2} = \sqrt{a^2 + r^2 - 2ax},$$

and
$$\frac{d\delta}{dx} = \frac{-a}{\sqrt{a^2 + r^2 - 2ax}} < 0.$$

Hence δ is a decreasing function in $\mathfrak{A} = (-r, r)$, and has no maximum or minimum at any point in $\mathfrak{A}$. It has, however, a maximum and a minimum in $\mathfrak{A}$, viz.
$$\text{Max } \delta = \overline{AC}, \quad \text{Min } \delta = \overline{AB}.$$

SEVERAL VARIABLES

Definite and Indefinite Forms

476. **1.** Let $f(x_1 \cdots x_m)$ be defined over a region, of which a is a point.

If
$$\Delta f = f(x) - f(a) > 0, \qquad \text{in } D^*(a),$$
f has a *minimum at* a. If
$$\Delta f < 0, \qquad \text{in } D^*(a),$$
f has a *maximum at* a.

The theorem of 468 may be generalized thus:

Let the partial derivatives of $f(x_1 \cdots x_m)$ *of order* $n+1$ *be continuous in* $D(a)$. *Let the partial derivatives of order* $<n$ *vanish at* a, *while the derivatives of order* n *do not all vanish at* a. *Then if* n *is odd,* f *has no extreme at* a.

Let n *be even. If* $d^n f(a) > 0$ *in* $D^*(a)$, f *has a minimum; if it is* <0, *it has a maximum at* a. *If* $d^n f(a)$ *has both signs in* $D^*(a)$, f *has no extreme at* a.

Let $x_1 = a_1 + h_1 \cdots x_m = a_m + h_m$. Then
$$\Delta f = \frac{1}{n!} d^n f(a) + \frac{1}{n+1!} d^{n+1} f(a + \theta h),$$
by 434.

Let $\eta_1 \cdots \eta_m$ be the direction cosines [244, 4] of the line L joining a and x. Then
$$h_1 = r\eta_1 \cdots h_m = r\eta_m,$$
where
$$r = \sqrt{h_1^2 + \cdots + h_m^2}; \quad \eta_1^2 + \cdots + \eta_m^2 = 1.$$

Then
$$\Delta f = r^n H(\eta) + r^{n+1} K(\eta),$$
since $d^n f$, $d^{n+1} f$ are homogeneous in $h_1 \cdots h_m$.

Since the derivatives of order $n+1$ are continuous in $D_\delta(a)$, there exists a positive number G, such that
$$|K| < G, \qquad \text{in } D_\delta(a). \tag{1}$$

If now $d^n f(a) > 0$ in $D^*(a)$, H is > 0 on the sphere

$$\eta_1^2 + \cdots + \eta_m^2 = 1, \qquad (S$$

to which η is restricted.

Then, by 355, 2, there exists a $\lambda > 0$, such that

$$H > \lambda.$$

Then, by virtue of 1), we can choose $\delta' < \delta$ so small that

$$\Delta f > 0, \qquad \text{in } D_{\delta'}^*(a).$$

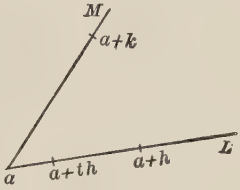
Hence f has a minimum at a.

Similar reasoning shows that if $d^n f(a) < 0$, f has a maximum.

Consider now the case that $d^n f(a)$ has both signs in $D^*(a)$. Suppose that it is positive at $a + h$ and negative at $a + k$.

Let L, M be the lines joining these points with a. Let η, κ be their direction cosines. Let

$$H(\eta) = A, \qquad A > 0.$$
$$H(\kappa) = B, \qquad B < 0.$$



Let $a + th$, $0 < t < 1$, be a point on L between a and $a + h$.

Let $$\text{Dist}(a, a+h) = r; \quad \text{then Dist}(a, a+th) = tr.$$

Thus $$\Delta f = f(a+th) - f(a) = t^n r^n \{A + trK'\}.$$

Let t_0 be such that $$t_0 r G < A.$$

Then, by 1), $\Delta f > 0$ for all points on L between a and $a + t_0 h$, a excluded. Similar reasoning shows that $\Delta f < 0$ for all points on M sufficiently near a, the point a excluded.

Thus in any domain of a, however small, Δf has opposite signs. Hence in this case f has no extreme at a.

Let n be odd. Then H being homogeneous,

$$H(-\eta) = (-1)^n H(\eta) = -H(\eta).$$

Hence $d^n f(a)$ has opposite signs in every domain of a. Hence when n is odd, f has no extreme at a.

2. *Let $f(x_1 \cdots x_m)$ have partial derivatives of the first order, finite or infinite, in the region R.*

The points of R, at which f has an extreme, satisfy the system of equations

$$\frac{\partial f}{\partial x_1} = 0 \quad \cdots \quad \frac{\partial f}{\partial x_m} = 0.$$

The demonstration is analogous to that of 468, 2.

477. 1. We have just seen that the sign of $d^n f(a)$ plays a decisive rôle in questions of maxima and minima. But as already observed, $d^n f$ is a homogeneous integral rational function of h_1, h_2, $\cdots$ h_m of degree n. Such functions are subjects of study in algebra and the theory of numbers, where they are often called *forms*.

A form $\Phi(x_1 \cdots x_m)$ which has always one sign, except at the origin where it necessarily vanishes, is called *definite*.

Such a form is

$$a_1^2 x_1^2 + \cdots + a_m^2 x_m^2, \tag{1}$$

the a's being not all 0.

If the sign of a definite form is positive, it is called a *positive definite form*; if negative, it is a *negative definite form*. Thus 1) is a positive definite form, while

$$-a_1^2 x_1^2 - \cdots - a_m^2 x_m^2$$

is an example of a negative definite form.

If Φ can take on both signs, it is called *indefinite*.

Thus

$$x_1^3 + \cdots + x_m^3$$

is an indefinite form.

There is a class of forms which vanish at points besides the origin and yet, when not 0, have always one sign. They are called *semidefinite* forms.

Such a form is, for example,

$$(a_1 x_1 + \cdots + a_m x_m)^2,$$

which is positive when not 0.

Consider the quadratic form

$$F = Ax^2 + 2\,Bxy + Cy^2.$$

If $A \neq 0$, we can write it

$$F = \frac{1}{A}\left\{ (Ax + By)^2 + (AC - B^2)y^2 \right\}.$$

If the determinant

$$D = AC - B^2$$

is > 0, F does not vanish except at the origin, and is therefore a positive definite form if $A > 0$, and a negative definite form if $A < 0$.

If $D < 0$, F is an indefinite form.

If $D = 0$,

$$F = \frac{1}{A}\left(Ax + By\right)^2.$$

Hence F vanishes on the line

$$Ax + By = 0,$$

but has otherwise one sign. Thus, in this case, F is semidefinite.

2. The theorem of 476 may now be stated as follows: *If $d^n f(a)$ is an indefinite form, f has no extreme at a. If it is a positive definite form, f has a minimum; if it is a negative definite form, f has a maximum at a.*

478. When $n = 2$, *i.e.* when not all partial derivatives of the second order are 0 at a, $d^n f(a)$ becomes a *quadratic form*,

$$d^2 f(a) = \Sigma a_{\iota\kappa} h_\iota h_\kappa; \qquad \iota, \kappa = 1, 2, \cdots m. \tag{1}$$

where

$$a_{\iota\kappa} = f''_{x_\iota x_\kappa}(a_1 \cdots a_m),$$

and hence

$$a_{\iota\kappa} = a_{\kappa\iota}.$$

The determinant

$$\Delta_m = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ \cdot & \cdot & \cdot & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mm} \end{vmatrix}$$

is called the *determinant of the form* 1).

Let Δ_{m-1} be obtained by deleting the last row and column in Δ_m; let Δ_{m-2} be obtained by deleting the last two rows and columns in Δ, etc.; finally, let $\Delta_0 = 1$.

In algebra the following theorem is proved:

In order that the form

$$\sum_{\iota, \kappa} a_{\iota\kappa} h_\iota h_\kappa \tag{2}$$

be a positive definite form, it is necessary and sufficient that the signs of

$$\Delta_0, \; \Delta_1, \; \cdots \; \Delta_m \tag{3}$$

are all positive. For 2) to be a definite negative form, it is necessary and sufficient that the signs in 3) are alternately positive and negative.

Applying this result to the theorem in 476, we have:

Let the partial derivatives of the third order be continuous in $D(a)$, and let those of the second order not all vanish at a. Let all the first derivatives vanish at a. Let

$$a_{\iota\kappa} = f''_{x_\iota x_\kappa}(a_1 \cdots a_m),$$

$$\Delta_r = \begin{vmatrix} a_{11} & \cdots & a_{1r} \\ \vdots & & \\ a_{r1} & \cdots & a_{rr} \end{vmatrix}. \qquad \begin{matrix} r = 0, 1, \cdots m. \\ \Delta_0 = 1. \end{matrix}$$

If the signs in the sequence

$$\Delta_0, \Delta_1, \cdots \Delta_m$$

are all positive, f has a minimum. If the signs in this sequence are alternately positive and negative, f has a maximum at a.

Semidefinite Forms

479. Up to the present, the case that $d^n f(a)$ is a semidefinite form has not been treated. It is, however, easy to show that in this case f may or may not have an extreme at a.

Ex. 1.

$$f(xy) = x^2 - 6\,xy^2 + 8\,y^4 = (x - 2\,y^2)(x - 4\,y^2).$$

$$a = (0, 0).$$

Here

$$f'_x(0) = 0, \quad f'_y(0) = 0;$$

$$f''_{x^2}(0) = 2, \quad f''_{xy}(0) = 0, \quad f''_{y^2}(0) = 0.$$

Hence,

$$d^2f(0) = h^2.$$

We have here a semidefinite form.

That f has not an extreme at the origin is obvious.

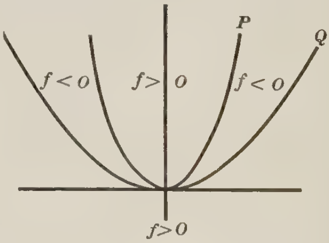
For, if P is the parabola

$$x = 2\,y^2,$$

and Q the parabola

$$x = 4\,y^2,$$

we see that $f < 0$ between these parabolas, and > 0 in the rest of the plane. points on the parabolas excepted, as in the figure.



Ex. 2.

$$f(xy) = y^2 + x^2y + x^4$$

$$= \left(y + \frac{x^2}{2}\right)^2 + \frac{3}{4}x^4. \qquad (1$$

$$a = (0, 0).$$

Obviously, from 1), f has a minimum at the origin.

Here

$$f_x'(0) = 0, \quad f_y'(0) = 0.$$

$$f_{x^2}''(0) = 0, \quad f_{xy}''(0) = 0, \quad f_{y^2}''(0) = 2.$$

Hence,

$$d^2f(0) = k^2.$$

We have here a semidefinite form.

480. It is beyond the scope of this work to do more than show that the semidefinite case is ambiguous and requires further investigation. We refer the reader for a detailed treatment of this case to Stolz, *Grundzüge*, Vol. 1, p. 211 *seq.*; Jordan, *Cours.*, Vol. 1, p. 380 *seq.*; Scheeffer, *Math. Ann.*, Vol. 35, p. 541; v. Dantscher, *Math. Ann.*, Vol. 42, p. 89.

Criticism

481. The partial derivatives of order n being continuous in $D(a)$, we have seen that

$$\Delta f = df(a) + \frac{1}{2!}d^2f(a) + \cdots + \frac{1}{n!}d^nf(a + \theta h) = T_1 + T_2 + \cdots + T_n.$$

The terms $T_1, T_2, \cdots$ are polynomials in $h_1, h_2, \cdots h_m$ of 1°, 2°, ⋯ degree, whose coefficients, except the last, are constant. Letting $h_1, h_2, \cdots$ be infinitesimals of the 1° order, T_r if $\neq 0$ is thus an infinitesimal of rth order. The assumption is now made by many authors that *if $r < s$, then T_s is infinitely small compared with T_r.* When there is only one variable h, this is indeed true; it is not, however, always true when there are two or more variables $h_1, h_2, \cdots$

As an example, consider the form

$$h_1^2 - 6\,h_1h_2^2 + 8\,h_2^4 = T_2 + T_3 + T_4.$$

Let the increments h_1, h_2 be related by the equation

$$h_2^2 = h_1; \tag{1}$$

i.e. let the point (h_1, h_2) approach the origin along the parabola 1)

Then

$$T_2 = h_2^4, \quad T_3 = -6\, h_2^4, \quad T_4 = 8\, h_2^4;$$

which shows that T_3, T_4, instead of being infinitely small compared with T_2, are in fact numerically 6 and 8 times larger than T_2.

482. Let us see now how this erroneous assumption regarding infinitesimals, when applied to the semidefinite case, leads to a false result.* For simplicity we take only two variables x, y.

Let

$$\Delta f = f(a+h,\, b+k) - f(a,\, b)$$
$$= \frac{1}{2!}\{Ah^2 + 2\,Bhk + Ck^2\} + \cdots$$
$$= T_2 + T_3 + \cdots$$

Let the determinant $AC - B^2 = 0$. Then T_2 is a semidefinite form. To fix the ideas let $A \neq 0$; then

$$T_2 = \frac{1}{2\,A}\{Ah + Bk\}^2, \quad \text{by } \mathbf{477,\ 1.}$$

Thus Δf has the sign of A, except for the points (h, k) on the line L,

$$Ax + By = 0. \tag{L}$$

For points on L, Δf becomes

$$T_3 + T_4 + \cdots$$

For points on the line L, on opposite sides of the origin, T_3 takes on opposite signs. As the sign of Δf at these points depends on the sign of T_3 (making use of the above erroneous assumption), it is thus necessary that $T_3 = 0$ for points on L, if f is to have an extreme at a, b. If $T_3 = 0$ for these points,

$$\Delta f = T_4 + \cdots$$

---

* Cf. Todhunter, *Differential Calculus*; Desmartes, *Cours d'Analyse.*

for these points. If now $T_4 \neq 0$ on L, f has an extreme* at (a, b), if T_4 has the same sign as A, for points of L, the origin excluded.

That this result is wrong may be shown by applying it to

$$f(xy) = x^2 - 6\,xy^2 + 8\,y^4,$$

which we considered in 479, Ex. 1.

Here

$$T_2 = h^2,\ \ T_3 = -6\,hk^2,\ \ T_4 = 8\,k^4, \qquad A = 2.$$

The line L is, in this case, the y-axis. For points on L, $T_3 = 0$; while $T_4 > 0$, the origin excepted. Thus T_4 has the same sign as A. We should have therefore an extreme at the origin, if the above reasoning were correct. But as we already saw in 479, f has no extreme at the origin.

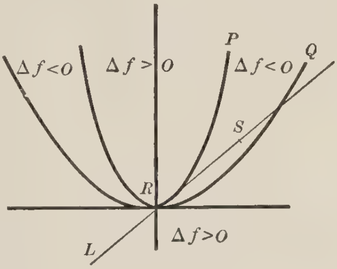
483. Another error which is sometimes made is the following. It is assumed that the function $f(x, y)$ has an extreme at the point R when and only when f has an extreme along every right line through R.

That this view is incorrect is seen by the function

$$f(xy) = (x - 2\,y^2)(x - 4\,y^2),$$

given in 479, Ex. 1.

As the figure shows, a point S moving along any line L toward the origin R, finally remains in a region for which $\Delta f > 0$, the origin of course excluded. If, therefore, this view were correct, f would have a minimum at R, which we know is not true.



Relative Extremes

484. 1. Let us consider the problem of finding the points of maxima and minima of a function

$$w = f(x_1 \cdots x_m,\ u_1 \cdots u_p); \tag{1}$$

---

* According to the above erroneous hypothesis.

where the variables $u_1 \cdots u_p$ are one-valued functions of $x_1 \cdots x_m$, defined over a region R and satisfying the system

$$\begin{aligned} \phi_1(x_1 \cdots u_p) &= 0 \\ \cdot \quad & \cdot \\ \phi_p(x_1 \cdots u_p) &= 0. \end{aligned} \tag{2}$$

Such points of maxima and minima are called points of *relative extreme*, to distinguish them from the case when the variables $x_1 \cdots x_m, u_1 \cdots u_p$ are all independent.

Let the point $(x_1 \cdots u_p)$ run over the region T when $(x_1 \cdots x_m)$ runs over R. Let $f, \phi_1 \cdots \phi_p$ have continuous first partial derivatives with respect to $x_1 \cdots u_p$ in T, and let the u's have continuous first partial derivatives with respect to $x_1 \cdots x_m$ in R.

Let w considered as a function of $x_1 \cdots x_m$ be denoted by

$$w = F(x_1 \cdots x_m).$$

The points of extreme of w in R satisfy the system

$$\frac{\partial F}{\partial x_1} = 0 \cdots \frac{\partial F}{\partial x_m} = 0, \tag{3}$$

by 476, 2. Let S denote the set of points determined by 3). Lagrange has given a method for forming the system 3), which is often serviceable. It rests on the introduction of certain *undetermined multipliers* $\mu_1 \cdots \mu_p$. In fact, differentiating 1) 2) we get:

$$dw = \frac{\partial f}{\partial x_1} dx_1 + \cdots + \frac{\partial f}{\partial x_m} dx_m + \frac{\partial f}{\partial u_1} du_1 + \cdots + \frac{\partial f}{\partial u_p} du_p ;$$

$$d\phi_1 = \frac{\partial \phi_1}{\partial x_1} dx_1 + \cdots + \frac{\partial \phi_1}{\partial x_m} dx_m + \frac{\partial \phi_1}{\partial u_1} du_1 + \cdots + \frac{\partial \phi_1}{\partial u_p} du_p = 0 ;$$

$$\vdots$$

$$d\phi_p = \frac{\partial \phi_p}{\partial x_1} dx_1 + \cdots + \frac{\partial \phi_p}{\partial x_m} dx_m + \frac{\partial \phi_p}{\partial u_1} du_1 + \cdots + \frac{\partial \phi_p}{\partial u_p} du_p = 0.$$

Multiply the 2d, 3d, ⋯ equations by $\mu_1, \mu_2, \cdots$ respectively, and add the results to the first equation.

We get

$$dw = \left(\frac{\partial f}{\partial x_1} + \sum_r \mu_r \frac{\partial \phi_r}{\partial x_1}\right) dx_1 + \cdots + \left(\frac{\partial f}{\partial u_p} + \sum_r \mu_r \frac{\partial \phi_r}{\partial u_p}\right) du_p. \tag{4}$$

Let us now, if possible, determine the μ's so that

$$\frac{\partial f}{\partial u_1}+\sum\mu_r\frac{\partial\phi_r}{\partial u_1}=0,\ \cdots\ \frac{\partial f}{\partial u_p}+\sum\mu_r\frac{\partial\phi_r}{\partial u_p}=0. \tag{5}$$

Then 4) gives

$$dw=\left(\frac{\partial f}{\partial x_1}+\sum\mu_r\frac{\partial\phi_r}{\partial x_1}\right)dx_1+\cdots+\left(\frac{\partial f}{\partial x_m}+\sum\mu_r\frac{\partial\phi_r}{\partial x_m}\right)dx_m. \tag{6}$$

Then, by 429, 2,

$$\frac{\partial F}{\partial x_1}=\frac{\partial f}{\partial x_1}+\sum\mu_r\frac{\partial\phi_r}{\partial x_1},\ \cdots\ \frac{\partial F}{\partial x_m}=\frac{\partial f}{\partial x_m}+\sum\mu_r\frac{\partial\phi_r}{\partial x_m}.$$

These are the left hand members of the equations 3). Hence, by 3),

$$\frac{\partial f}{\partial x_1}+\sum\mu_r\frac{\partial\phi_r}{\partial x_1}=0,\ \cdots\ \frac{\partial f}{\partial x_m}+\sum\mu_r\frac{\partial\phi_r}{\partial x_m}=0. \tag{7}$$

Thus the points of S are determined by 2), 5), 7).

2. Let us introduce the function

$$g=f+\mu_1\phi_1+\cdots+\mu_p\phi_p.$$

We observe that, considering $x_1\cdots u_p$ as independent variables,

$$\frac{\partial g}{\partial x_1}=0,\ \cdots\ \frac{\partial g}{\partial x_m}=0,\qquad \frac{\partial g}{\partial u_1}=0,\ \cdots\ \frac{\partial g}{\partial u_p}=0 \tag{8}$$

are precisely the equations 5), 7).

485. To determine whether a point of S is a point of extreme, it is often necessary to consider the second and even higher differentials of $F(x_1\cdots x_m)$. Here it is sometimes convenient to make use of the fact that

$$d^2F=d^2g\,;$$

where the differential on the right is calculated supposing

$$x_1\cdots x_m,\ u_1\cdots u_p$$

to be independent variables.

For, let us denote g considered as a composite function of x by $G(x_1 \cdots x_m)$. Then

$$F(x_1 \cdots x_m) = G(x_1 \cdots x_m),$$

since the ϕ's vanish now by 484, 2.

Hence

$$d^2F = d^2G.$$

But, by 433, 2),

$$d^2G = d^2g + \sum \frac{\partial g}{\partial x_\iota} d^2x_\iota + \sum \frac{\partial g}{\partial u_\iota} d^2u_\iota$$

$$= d^2g, \text{ by 484, 8)}.$$

486. *Example.* Let us find the shortest distance from the point $P = (a_1 a_2 a_3)$, to the plane

$$\phi = b_1x_1 + b_2x_2 + b_3x_3 + b_0 = 0. \qquad (1$$

Let

$$w = \delta^2 = \sum_\iota (x_\iota - a_\iota)^2 \qquad \iota = 1, 2, 3.$$

$$= f(x_1x_2x_3).$$

The points of minimum value are the same for w and δ.

We have

$$g = f + \mu\phi,$$

$$\frac{\partial g}{\partial x_\iota} = 2(x_\iota - a_\iota) + \mu b_\iota = 0. \qquad \iota = 1, 2, 3. \qquad (2$$

From the four equations 1), 2), we find

$$x_\iota - a_\iota = -\tfrac{1}{2}\mu b_\iota,$$

$$\mu = 2\frac{a_1b_1 + a_2b_2 + a_3b_3 + b_0}{b_1^2 + b_2^2 + b_3^2}.$$

Hence at this point $x = \xi$,

$$w = \frac{(a_1b_1 + a_2b_2 + a_3b_3 + b_0)^2}{b_1^2 + b_2^2 + b_3^2}. \qquad (3$$

To ascertain if ξ is a point of minimum, consider the value of d^2w at this point.

We have

$$dg = \Sigma\{2(x_\iota - a_\iota) + \mu b_\iota\} dx_\iota, \qquad \iota = 1, 2, 3.$$

$$d^2g = 2\sum_\iota dx_\iota^2.$$

As this is positive, ξ is a point of minimum. Thus the least distance δ_0 from a to ϕ is determined by 3). We get

$$\delta_0 = \frac{a_1b_1 + a_2b_2 + a_3b_3 + b_0}{\sqrt{b_1^2 + b_2^2 + b_3^2}}.$$

CHAPTER XII

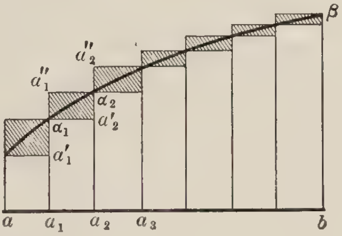
INTEGRATION

Geometric Orientation

487. 1. As the reader is probably aware, the integral calculus arose from attempts to find the length of curves, the area of surfaces, and the volume of solids.

Before taking up the general theory of integration, let us see how the problem of finding the area of a simple figure leads to an integral.

2. In the interval $\mathfrak{A} = (a, b)$ let $y = f(x)$ be an increasing continuous function whose graph Γ is given in the adjoining figure.



We seek the area A of the figure $ab\alpha\beta = F$. The upper boundary of F is the curve Γ.

To find the area of a circle in elementary geometry, we form a sequence of inscribed and circumscribed polygons. Each inscribed polygon is contained in the circle; each circumscribed polygon contains the circle. We say the area of the circle is therefore less than any of the outer polygons, and greater than any of the inner polygons. We then show that the areas of these two systems of polygons have a common limit as the number of the sides increases indefinitely. This limit is then, by definition, the area of the circle.

We shall adopt a similar procedure here, reserving for later a more thorough discussion in connection with other fundamental geometric notions.

Let us divide (a, b) into n equal intervals by introducing the points $a_1, a_2, a_3, \cdots$

Over each interval (a_r, a_{r+1}) we have two rectangles

$$r_m = a_m a_{m+1} \alpha_m a'_{m+1}, \qquad \alpha_m = f(a_m).$$

and

$$R_m = a_m a_{m+1} a''_m \alpha_{m+1}.$$

Let

$$s_n = r_0 + r_1 + \cdots + r_{n-1},$$

$$S_n = R_0 + R_1 + \cdots + R_{n-1}.$$

Then S_n contains F, while s_n is contained in F.

Let us now divide each of the intervals

$$(a, a_1), \ (a_1, a_2), \ \cdots (a_{n-1}, b)$$

into two equal parts. We get two new sums s_{2n} and S_{2n}. In this way we may continue without end. Let us now give n the values 1, 2, 4, ⋯

We get two limited univariant sequences

$$s_1 < s_2 < s_4 < s_8 \cdots < S_1;$$

$$S_1 > S_2 > S_4 > S_8 \cdots > s_1.$$

Each sequence has a limit by 109. These limits are, moreover, the same. For $S_n - s_n$ is obviously the area of the shaded region in the figure.

Hence

$$S_n - s_n = (\beta - \alpha)\frac{b-a}{n}.$$

Evidently

$$\lim_{n=\infty} (S_n - s_n) = 0.$$

Hence

$$\lim S_n = \lim s_n.$$

As in the case of the circle, the common limit is, by definition, the area of F.

3. Now

$$r_m = \frac{b-a}{n} f(a_m).$$

Hence

$$s_n = \frac{b-a}{n}\{f(a) + f(a_1) + \cdots + f(a_{n-1})\}; \qquad (1$$

and therefore, setting for uniformity, $a_0 = a$,

$$A = \lim_{n=\infty} \sum_{m=0}^{n-1} f(a_m) \frac{b-a}{n}. \qquad (2$$

But as the reader knows, the expression on the right of 2) is the integral

$$\int_a^b f(x)dx.$$

488. *Example.* Let us find the limit A of 487, 2) for the function

$$f(x) = c^x. \qquad c > 0.$$

Set

$$\frac{b-a}{n} = \delta\,;$$

then

$$a_m = a + m\delta, \qquad m = 0, 1, \cdots n-1.$$

and

$$f(a_m) = c^{a+m\delta} = c^a c^{m\delta}.$$

Hence

$$s_n = \delta \cdot c^a \{1 + c^\delta + c^{2\delta} + \cdots + c^{(n-1)\delta}\}$$

$$= \delta \cdot c^a \frac{1 - c^{n\delta}}{1 - c^\delta} = (c^b - c^a)\, \frac{\delta}{c^\delta - 1}. \qquad (1$$

Now

$$\lim_{n=\infty} \delta = 0.$$

Also, by 311,

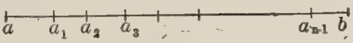
$$\lim_{\delta=0} \frac{\delta}{c^\delta - 1} = \frac{1}{\log c}. \qquad (2$$

From 1), 2) we have

$$A = \lim s_n = \frac{c^b - c^a}{\log c}.$$

Analytical Definition of an Integral

489. 1. Let $f(x)$ be a limited function, defined over the interval $\mathfrak{A} = (a, b)$, $a < b$.



Let us divide $\mathfrak{A}$ into n sub-intervals

$$\delta_m = (a_{m-1}a_m), \qquad m = 1, 2, \cdots n$$

by interpolating *at pleasure* the points

$$a_1,\ a_2,\ \cdots\ a_{n-1}. \qquad (1$$

For uniformity of notation, we set

$$a = a_0, \quad b = a_n.$$

This set of points 1) produces a division of $\mathfrak{A}$, which we denote by

$$D = D(a_1,\ a_2,\ \cdots\ a_{n-1}).$$

Since no confusion can arise, let δ_m denote also the length of the interval δ_m. The greatest of these lengths we call the *norm of D* and denote it by δ.

In each δ_m, let us take a point ξ_m at pleasure, and build the sum

$$J_\delta = \sum_{m=1}^{n} f(\xi_m)(a_m - a_{m-1}) = \sum_{D} f(\xi_m)\delta_m. \tag{2}$$

In passing, let us note that the sums S_n, s_n of 487 are special cases of 2).

Let now $\delta \doteq 0$. If J_δ converges to a limit J, which is independent of the choice of the points a_m, ξ_m, we write

$$J = \lim_{\delta=0} \sum f(\xi_m)\delta_m = \int_a^b f(x)dx.$$

We say *$f(x)$ is integrable from a to b*, and call J *the integral of $f(x)$ from a to b*. $f(x)$ is called the *integrand; a, b* are respectively *the lower and upper limits of integration.* We also write

$$J = \int_{\mathfrak{A}} f dx.$$

The symbol $\int$ is a long S, the first letter of the word *sum.*

2. To fix the ideas, we have taken $a < b$.

Then in 2), the numbers

$$\delta_m = (a_m - a_{m-1})$$

are positive.

If we had taken $a > b$, the δ's would be all negative. Evidently, whether a is greater or less than b, if

$$\int_a^b f(x)dx \tag{3}$$

exists, then

$$\int_b^a f(x)dx \tag{4}$$

exists and 3), 4) have the same numerical value, but are of opposite sign.

Without loss of generality, we may therefore, in our discussion, take $a < b$ so that the δ's are > 0.

3. Obviously the symbol

$$\int_a^b f(x)dx$$

has no sense when $a = b$. In this case we shall assign to it the value 0.

4. *Let* $f(x)$ *be integrable in* $\mathfrak{A} = (a, b)$, *and*

$$|f(x)| \leq M.$$

Then

$$\left|\int_a^b f dx\right| \leq M(b - a). \tag{5}$$

For,

$$-M \leq f(\xi_m) \leq M.$$

Hence

$$-\Sigma M\delta_m \leq J_\delta \leq \Sigma M\delta_m,$$

or,

$$-M(b - a) \leq J_\delta \leq M(b - a).$$

Passing to the limit, $\delta = 0$,

$$-M(b - a) \leq \int_a^b f dx \leq M(b - a),$$

which is 5).

Upper and Lower Integrals

490. Before deducing criteria for the integrability of $f(x)$, we define *upper* and *lower integrals.*

Let

$$M_\kappa = \text{Max} f(x), \qquad m_\kappa = \text{Min} f(x), \qquad \text{in } \delta_\kappa.$$

$$\overline{S}_D = \Sigma M_\kappa \delta_\kappa, \qquad \underline{S}_D = \Sigma m_\kappa \delta_\kappa.$$

We shall show immediately that the limits

$$\overline{S} = \lim_{\delta=0} \overline{S}_D, \qquad \underline{S} = \lim_{\delta=0} \underline{S}_D$$

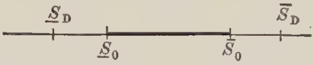
exist and are finite. They are called respectively the upper and lower integrals of $f(x)$ for the interval $\mathfrak{A}$, and are denoted by

$$\overline{\int_{\mathfrak{A}}} f dx, \qquad \underline{\int_{\mathfrak{A}}} f dx.$$

491. *If $f(x)$ is limited in $\mathfrak{A}$, the upper and lower integrals exist and are finite.*

Let us consider the upper integral; similar reasoning is applicable to the lower integral.

Corresponding to each division D there is a $\overline{S}_D$. Let us lay off these values on an axis. We get a limited point aggregate.



For, since $f(x)$ is limited, there exists a number $M>0$, such that

$$-M \leq f(x) \leq M.$$

Hence

$$-\Sigma M\delta_\kappa \leq \Sigma M_\kappa \delta_\kappa \leq \Sigma M\delta_\kappa,$$

or

$$-M(b-a) \leq \overline{S}_D \leq M(b-a).$$

Hence, the $\overline{S}_D$ are limited.

Let

$$\overline{S}_0 = \text{Min } \overline{S}_D.$$

We show now that

$$\overline{S}_0 = \lim_{\delta=0} \overline{S}_D;$$

that is, for each $\epsilon > 0$, there exists a δ_0, such that

$$\overline{S}_D - \overline{S}_0 < \epsilon \tag{1}$$

for any division D of norm $\delta \leq \delta_0$.

Since $\overline{S}_0$ is a minimum, there exists a division Δ of $\mathfrak{A}$ of norm η, such that

$$\overline{S}_0 \leq \overline{S}_\Delta < \overline{S}_0 + \frac{\epsilon}{2}. \tag{2}$$

Let $\eta_1, \eta_2, \cdots \eta_\nu$ be the intervals of Δ. Let

$$\delta_{\iota_1}, \ \delta_{\iota_2}, \ \delta_{\iota_3}, \ \cdots$$

be the intervals of D lying wholly in η_ι, $\iota = 1, 2, \cdots \nu$; let

$$\delta_1', \ \delta_2', \ \cdots$$

be the other intervals of D.

We take now δ_0 so small that

$$\sum \delta_\iota' < \frac{\epsilon}{4M}, \qquad \sum_\iota \Big(\eta_\iota - \sum_\kappa \delta_{\iota\kappa}\Big) < \frac{\epsilon}{4M} \tag{3}$$

for all $\delta < \delta_0$. This is evidently possible, since Δ has only ν intervals, ν being fixed.

Let

$$M_{\iota\kappa} = \text{Max} f, \text{ in } \delta_{\iota\kappa},$$

$$M_\iota' = \text{Max} f, \text{ in } \delta_\iota'.$$

Then

$$\begin{aligned}\overline{S}_D &= \Sigma M_{\iota\kappa} \delta_{\iota\nu} + \Sigma M_\iota' \delta_\iota' \\ &\leq \Sigma M_\iota \delta_{\iota\kappa} + M \Sigma \delta_\iota' \\ &\leq \overline{S}_\Delta - \overline{S}_\Delta + \Sigma M_\iota \delta_{\iota\kappa} + M \Sigma \delta_\iota' \\ &\leq \overline{S}_\Delta + M \Sigma \delta_\iota' - (\Sigma M_\iota \eta_\iota - \Sigma M_\iota \delta_{\iota\kappa}) \\ &\leq \overline{S}_\Delta + M \Sigma \delta_\iota' + M(\Sigma \eta_\iota - \Sigma \delta_{\iota\kappa}) \\ &< \overline{S}_\Delta + \frac{\epsilon}{2}, \text{ by 3)}. \end{aligned} \tag{4}$$

Hence, from 2), 4),

$$\overline{S}_0 \leq \overline{S}_D < \overline{S}_\Delta + \frac{\epsilon}{2} < \overline{S}_0 + \epsilon\,;$$

or,

$$\overline{S}_D - \overline{S}_0 < \epsilon, \qquad \delta < \delta_0. \qquad \text{Q. E. D}$$

492. Ex. 1.

$$\mathfrak{A} = (0, 1).$$

$$f(x) = 1 \text{ for rational points in } \mathfrak{A},$$

$$= 0 \text{ for irrational points in } \mathfrak{A}.$$

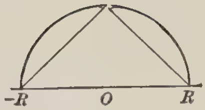
Then

$$\overline{S}_D = 1, \qquad \underline{S}_D = 0.$$

$$\overline{S} = 1, \qquad \underline{S} = 0.$$

Ex. 2.

$$\mathfrak{A} = (-R, R).$$



Let $f(x)$ lie on the circumference of the circle for rational x, while it lies on the edge of the inscribed square for irrational x.

Then evidently

$$\overline{\int}_{\mathfrak{A}} fdx = \tfrac{1}{2}\pi R^2, \text{ area of semicircle ;}$$

$$\underline{\int}_{\mathfrak{A}} fdx = R^2, \text{ area of half the inscribed square.}$$

Criteria for Integrability

493. *For the limited function $f(x)$ to be integrable in the interval $\mathfrak{A}$, it is necessary and sufficient that*

$$\overline{\int}_{\mathfrak{A}} f dx = \underline{\int}_{\mathfrak{A}} f dx.$$

It is sufficient. For, let D be any division of norm δ.

Then

$$M_\kappa \geq f(\xi_\kappa) \geq m_\kappa.$$

Hence

$$M_\kappa \delta_\kappa \geq f(\xi_\kappa)\delta_\kappa \geq m_\kappa \delta_\kappa.$$

Summing, we get

$$\overline{S}_D \geq J_\delta \geq \underline{S}_D.$$

By hypothesis,

$$\lim_{\delta=0} \overline{S}_D = \lim_{\delta=0} \underline{S}_D = L, \text{ say.}$$

Hence

$$\lim J_\delta = L.$$

It is necessary. For, since the integral J exists,

$$\epsilon > 0, \quad \delta_0 > 0, \quad |J - \Sigma f(\xi_\kappa)\delta\,| < \frac{\epsilon}{4} \qquad (1$$

for any division D of norm $\delta < \delta_0$.

Let η_κ be any other point in the subinterval δ_κ belonging to the division D just mentioned.

We have also, as in 1),

$$|J - \Sigma f(\eta_\kappa)\delta_\kappa| < \frac{\epsilon}{4}. \qquad (2$$

Subtracting 1), 2), we have

$$|\Sigma f(\xi_\kappa)\delta_\kappa - \Sigma f(\eta_\kappa)\delta_\kappa| < \frac{\epsilon}{2}, \tag{3}$$

for any division of norm $\delta < \delta_0$.

On the other hand, in each δ_κ there are points ξ_κ, η_κ, such that

$$M_\kappa < f(\xi_\kappa) + \frac{\epsilon}{4(b-a)},$$

$$m_\kappa > f(\eta_\kappa) - \frac{\epsilon}{4(b-a)}.$$

Multiplying these inequalities by δ_κ and adding, we have

$$\overline{S}_D = \Sigma M_\kappa \delta_\kappa < \Sigma f(\xi_\kappa)\delta_\kappa + \frac{\epsilon}{4},$$

$$\underline{S}_D = \Sigma m_\kappa \delta_\kappa > \Sigma f(\eta_\kappa)\delta_\kappa - \frac{\epsilon}{4}.$$

Hence

$$\overline{S}_D - \underline{S}_D < \Sigma f(\xi_\kappa)\delta_\kappa - \Sigma f(\eta_\kappa)\delta_\kappa + \frac{\epsilon}{2}.$$

This gives, using 3),

$$\overline{S}_D - \underline{S}_D < \epsilon.$$

Hence

$$\overline{S} - \underline{S} < \epsilon ; \quad \text{hence } \overline{S} = \underline{S}.$$

494. We can state the theorem of 493 a little differently by introducing the following definitions. The difference between the maximum and minimum of a function $f(x)$ in an interval $\mathfrak{A}$, is called the *oscillation of f in* $\mathfrak{A}$. It cannot ever be negative. Let D be any division of $\mathfrak{A}$ into subintervals δ_κ, of length δ_κ. Let ω_κ be the oscillation of f in δ_κ. The sum

$$\Sigma \omega_\kappa \delta_\kappa = \Omega_D f = \Omega f$$

is called the *oscillatory sum* of f for the division D.

We have

$$\Omega_D f = \Sigma (M_\kappa - m_\kappa)\, \delta_\kappa = \Sigma M_\kappa \delta_\kappa - \Sigma m_\kappa \delta_\kappa$$

$$= \overline{S}_D - \underline{S}_D. \tag{1}$$

495. *In order that the limited function $f(x)$ be integrable in $\mathfrak{A}$, it is necessary and sufficient that*

$$\lim_{\delta=0} \Omega f = 0. \qquad (1$$

For, by 494, 1),

$$\Omega f = \overline{S}_D - \underline{S}_D.$$

By 493, $f(x)$ is integrable when and only when

$$\lim_{\delta=0} \overline{S}_D = \lim \underline{S}_D,$$

or when and only when

$$\lim_{\delta=0} (\overline{S}_D - \underline{S}_D) = 0,$$

which is 1).

496. *If $f(x)$ is integrable in $\mathfrak{A}$, it is integrable in any partial interval $\mathfrak{B}$ of $\mathfrak{A}$.*

Let $\mathfrak{A} = (a, b)$, $\mathfrak{B} = (\alpha, \beta)$.

Since

$$\lim_{\delta=0} \sum_D \omega_\kappa \delta_\kappa = 0 \qquad (1$$

for *any* system of divisions whose norm $\delta \doteq 0$, let us consider only such divisions involving the points α, β. Let D_1 be the division of $\mathfrak{B}$, produced by D. Then

$$0 \leq \sum_{D_1} \omega_\kappa \delta_\kappa \leq \sum_D \omega_\kappa \delta_\kappa, \qquad (2$$

since the first sum contains only a part of the intervals δ_κ, and ω_κ is $\geqq 0$.

Passing to the limit in 2), we have, by 1),

$$\lim_{\delta=0} \sum_D \omega_\kappa \delta_\kappa = 0.$$

Hence, $f(x)$ is integrable in $\mathfrak{B}$.

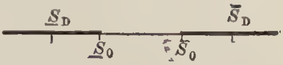
497. *In order that the limited function $f(x)$ be integrable in $\mathfrak{A}$, it is necessary and sufficient that, for each $\epsilon > 0$, there exists at least one division D for which*

$$\Omega_D f < \epsilon.$$

That this condition is *necessary* follows at once from 495.

It is sufficient. For, by 494, for the division D

$$\overline{S}_D - \underline{S}_D < \epsilon.$$



But then, as the figure shows,

$$\overline{S}_0 - \underline{S}_0 < \epsilon;$$

or, by 491,

$$\overline{\int}_{\mathfrak{A}} f dx - \underline{\int}_{\mathfrak{A}} f dx < \epsilon.$$

But then, by 87, 5,

$$\overline{\int}_{\mathfrak{A}} f dx = \underline{\int}_{\mathfrak{A}} f dx.$$

Therefore, by 493, $f(x)$ is integrable.

498. *In order that the limited function $f(x)$ be integrable in $\mathfrak{A}$, it is necessary and sufficient that for any pair of positive numbers ω, σ, there exists a division D of $\mathfrak{A}$, such that the sum of the subintervals** *of D in which the oscillation of $f(x)$ is $> \omega$, is $< \sigma$.*

It is necessary. For, by 497, there exists a division D for which $\Omega_D f$ is as small as we please, and therefore

$$\Omega_D f = \Sigma\omega_\kappa\delta_\kappa < \omega\sigma. \tag{1}$$

Let the intervals of D for which the oscillation of f is $\geqq \omega$, be denoted by D_κ, those for which the oscillation is $< \omega$, by d_κ.

Then
$$\Sigma\omega_\kappa\delta_\kappa = \Sigma\omega_\kappa D_\kappa + \Sigma\omega_\kappa d_\kappa \geqq \Sigma\omega_\kappa D_\kappa \geqq \omega\Sigma D_\kappa.$$

This, with 1), gives
$$\omega\sigma > \omega\Sigma D_\kappa.$$

Hence
$$\Sigma D_\kappa < \sigma.$$

It is sufficient. For, having taken $\epsilon > 0$ small at pleasure, take

$$\sigma = \frac{\epsilon}{2(M-m)}, \quad \omega = \frac{\epsilon}{2(b-a)}, \tag{2}$$

where

$$M = \text{Max}\, f, \ m = \text{Min}\, f, \qquad \text{in } \mathfrak{A}.$$

---

* For brevity, instead of *sum of the lengths of the subintervals.*

Then, by hypothesis, there exists a division D for which

$$\Sigma D_\kappa < \sigma.$$

This, with 2), gives

$$\begin{aligned}\Omega_D f &= \Sigma\omega_\kappa D_\kappa + \Sigma\omega_\kappa d_\kappa \\ &< (M-m)\Sigma D_\kappa + \omega\Sigma d_\kappa \\ &< (M-m)\sigma + \omega(b-a) = \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.\end{aligned}$$

There is, therefore, at least one division D for which

$$\Omega_D f < \epsilon.$$

Then, by 497, f is integrable in $\mathfrak{A}$.

Classes of Limited Integrable Functions

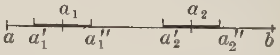
499. *If $f(x)$ is continuous in the interval $\mathfrak{A}$, it is integrable in $\mathfrak{A}$.*

For, since f is continuous in $\mathfrak{A}$, it is uniformly continuous. Hence, by 353, we can divide $\mathfrak{A}$ into subintervals of length $\delta > 0$, such that the oscillation of f in each interval is $< \omega$, an arbitrarily small positive number. There is thus no subinterval in which the oscillation $\geqq \omega$.

Therefore, by 498, f is integrable in $\mathfrak{A}$.

500. *If $f(x)$ is limited in the interval $\mathfrak{A} = (a, b)$ and has only a finite number of points of discontinuity $a_1, a_2, \cdots a_s$, it is integrable in $\mathfrak{A}$.*

Let ω, σ be any pair of positive numbers. On either side of the points a_κ mark the points a_κ', a_κ'', $\kappa = 1, 2, \cdots s$, as in the figure; but such that the total length of these little intervals is $< \sigma$.



Since f is continuous in (a, a_1'), we can divide it into subintervals such that the oscillation of f in each of them is $< \omega$.

The same is true of the intervals (a_1'', a_2'), (a_2'', a_3'), $\cdots$

But this set of subintervals in $\mathfrak{A}$ gives a division of $\mathfrak{A}$ for which the sum of the intervals in which the oscillation is $\geqq \omega$ is $< \sigma$. Hence, by 498, f is integrable in $\mathfrak{A}$.

501. 1. In 263 we saw there were point aggregates having derivatives (not 0 of course) of every order. This leads us to divide point aggregates into two *classes* or *species*. The first embraces all point aggregates whose derivatives after some order vanish. The second embraces aggregates having non-zero derivatives of every order.

2. *Let $f(x)$ be limited in the interval $\mathfrak{A}$. If its points of discontinuity form an aggregate Δ of the first species, f is integrable in $\mathfrak{A}$.*

We note that the aggregate Δ in 500 is of order 0.

We prove the above theorem by complete induction. Let us assume therefore that f is integrable when Δ is of order $n-1$, and show f is integrable when Δ is of order n.

Since Δ is of order n, $\Delta^{(n)}$ embraces only a finite number of points, by 265. Call these

$$a_1, \; a_2, \; \cdots \; a_s.$$

As in 500, we can inclose them in little intervals $(a_1', a_1'') \cdots$

The points of discontinuity in the intervals

$$\mathfrak{A}_1 = (a, a_1'), \;\; \mathfrak{A}_2 = (a_1'', a_2'), \;\; \cdots$$

form an aggregate of order $n-1$. Hence, by hypothesis, $f(x)$ is integrable in $\mathfrak{A}_1, \mathfrak{A}_2, \cdots$

Then each interval $\mathfrak{A}_\kappa$ can be divided into subintervals by 498, so that the sum of the intervals in which the oscillation of f is $> \omega$ is as small as we please. As in 500, the totality of these little subintervals furnishes a division of $\mathfrak{A}$ for which the sum of the intervals in which the oscillation is $> \omega$ is $< \sigma$, an arbitrarily small number. Then, by 498, $f(x)$ is integrable in $\mathfrak{A}$.

502. *Let $f(x)$ be limited and monotone in $\mathfrak{A}$; then $f(x)$ is integrable in $\mathfrak{A}$.*

If $f(x)$ is constant, the theorem is obvious. We may therefore exclude this case. We show that for each $\epsilon > 0$ there exists a division D for which

$$\Omega_D f < \epsilon.$$

Then, by 497, f is integrable.

To fix the ideas, suppose $f(x)$ is increasing. Let us divide $\mathfrak{A}$ into equal intervals of length

$$\delta < \frac{\epsilon}{f(b) - f(a)}. \tag{1}$$

Then

$$\begin{aligned}\Omega_D f &= \delta[\{f(a_1) - f(a)\} + \{f(a_2) - f(a_1)\} + \cdots + \{f(b) - f(a_{n-1})\}] \\ &= \delta\{f(b) - f(a)\} \\ &< \epsilon, \text{ by } 1).\end{aligned}$$

503. *Example.*

For

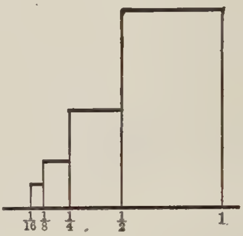
$$\frac{1}{2^{n+1}} < x \overline{>} \frac{1}{2^n},$$

let

$$f(x) = \frac{1}{2^n}. \qquad n = 0, 1, 2, 3, \cdots$$

Let

$$f(0) = 0.$$



Here f is monotone increasing and limited in the interval $\mathfrak{A} = (0, 1)$. It is therefore integrable in $\mathfrak{A}$. It has an infinite number of points of discontinuity, viz. the points

$$x = \frac{1}{2^n}. \qquad n = 1, 2, \cdots$$

Properties of Integrable Functions

504. *If $f_1(x), \cdots f_s(x)$ are limited and integrable in the interval $\mathfrak{A}$.*

Then $$F(x) = c_1 f_1 + \cdots + c_s f_s, \qquad c\text{'s constants},$$

is integrable in $\mathfrak{A}$, and

$$\int_{\mathfrak{A}} F dx = c_1 \int_{\mathfrak{A}} f_1 dx + \cdots + c_s \int_{\mathfrak{A}} f_s dx. \tag{1}$$

For,

$$\Sigma F(\xi_\kappa)\delta_\kappa = c_1 \Sigma f_1(\xi_\kappa)\delta_\kappa + \cdots + c_s \Sigma f_s(\xi_\kappa)\delta_\kappa.$$

Passing to the limit, we get 1).

505. *If $f(x)$, $g(x)$ are limited and integrable in $\mathfrak{A}$,*

$$h(x) = f(x) g(x)$$

is integrable in $\mathfrak{A}$.

1. Suppose first that $f(x)$, $g(x)$ are >0 and $<M$ in $\mathfrak{A}$. Let D be any division of $\mathfrak{A}$. Let δ_κ be one of the subintervals. Let

$$O_h, \ O_f, \ O_g$$

be the oscillations of $h(x)$, $f(x)$, $g(x)$ in δ_κ.

Let

$$F, \ f,$$

$$G, \ g,$$

be the maximum and minimum of $f(x)$ and $g(x)$ respectively in δ_κ.

Then

$$O_h \overset{=}{<} FG - fg = F(G-g) + g(F-f)$$

$$= FO_g + gO_f \leq MO_f + MO_g.$$

Hence

$$\Omega_D h \leq M\Sigma\, O_f \delta_\kappa + M\Sigma\, O_g \delta_\kappa = M\Omega_D f + M\Omega_D g.$$

Since for $\delta = 0$,

$$\lim \Omega_D f = 0, \ \lim \Omega_D g = 0,$$

we have

$$\lim \Omega_D h = 0.$$

Hence, by 495, $h(x)$ is integrable in $\mathfrak{A}$.

2. If $f(x)$, $g(x)$ are not positive, we can add the positive numbers α, β to them so that

$$f_1(x) = f(x) + \alpha, \ g_1(x) = g(x) + \beta$$

are positive, and by 504, integrable.

Then

$$f_1(x) g_1(x) = f(x)g(x) + \alpha g(x) + \beta f(x) + \alpha\beta.$$

Hence

$$f(x)g(x) = f_1(x)g_1(x) - \alpha g(x) - \beta f(x) - \alpha\beta. \tag{1}$$

Each term on the right side of 1) is integrable by what precedes. Hence $f(x)g(x)$ is.

506. *In the interval* $\mathfrak{A}$ *let*

$$A < f(x) < B,$$

where A, B *are both positive or both negative numbers. If* $f(x)$ *is integrable in* $\mathfrak{A}$, *so is*

$$g(x) = \frac{1}{f(x)}.$$

To fix the ideas, suppose A, B are positive. Using the notation of 505,

$$O_g = \frac{1}{f} - \frac{1}{F} = \frac{F-f}{fF} < \frac{F-f}{A^2} = \frac{O_f}{A^2}.$$

Hence

$$0 \leq \Omega_D g < \frac{1}{A^2}\,\Omega_D f. \tag{1}$$

As $f(x)$ is integrable,

$$\lim_{\delta=0} \Omega_D f = 0.$$

Hence $g(x)$ is also integrable, by 1) and 495.

507. *If* $f(x)$ *is limited and integrable in* $\mathfrak{A}$, *so is*

$$g(x) = |f(x)|.$$

For, using the notation of 505,

$$O_g \leq O_f$$

obviously. Therefore

$$0 \leq \Omega_D g \leq \Omega_D f.$$

As

$$\lim \Omega_D f = 0,$$

$g(x)$ is integrable by 495.

508. In 501, 503 we have met with integrable functions which have an infinite number of discontinuities in $\mathfrak{A}$. There is, however, a limit to the discontinuity of a function beyond which it ceases to be integrable, viz.:

If the limited function $f(x)$ is integrable in the interval $\mathfrak{A}$, there are an infinity of points in any partial interval $\mathfrak{B}$ of $\mathfrak{A}$, at which f is continuous.

Let $\omega_1 > \omega_2 > \cdots$ be positive and $\doteq 0$. By 498 there exists a division such that the sum of the intervals in which the oscillation of f is $\geq \omega_1$ is $< \sigma$. Thus, if σ be taken less than $\mathfrak{B}$, there is at least one subinterval within $\mathfrak{B}$, call it $\mathfrak{B}_1$, in which the oscillation is $< \omega_1$. Similarly, there is an interval $\mathfrak{B}_2$ within $\mathfrak{B}_1$, in which the oscillation of f is $< \omega_2$. Continuing this way, we can get a sequence of intervals

$$\mathfrak{B},\ \mathfrak{B}_1,\ \mathfrak{B}_2,\ \cdots \tag{1}$$

each contained within the preceding, whose lengths $\doteq 0$. Then, by 127, 2, the sequence 1) defines a point c within $\mathfrak{B}$, such that for every point x in $D(c)$,

$$|f(x) - f(c)| < \omega. \qquad \omega \textit{ arbitrarily small.}$$

Thus $\mathfrak{B}$ contains at least one point c, at which $f(x)$ is continuous. It therefore contains an infinity of such points.

Functions with Limited Variation

509. An important class of limited integrable functions is formed by functions with limited variation, which we now consider.

Let $f(x)$ be defined over the interval $\mathfrak{A} = (a, b)$.

Let D be a division of $\mathfrak{A}$ of norm δ; let $\delta_1, \delta_2, \cdots$ be the subintervals of $\mathfrak{A}$ corresponding to this division.

Let ω_κ denote the oscillation of $f(x)$ in δ_κ. Let us form the sum

$$\omega_D = \Sigma \omega_\kappa.$$

If

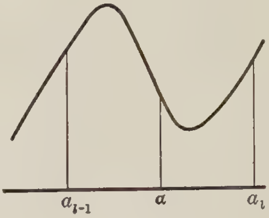
$$\tilde{\omega} = \text{Max } \omega_D$$

is finite for all possible divisions D, we say $f(x)$ is a function with *limited variation*, or that $f(x)$ has a limited variation in $\mathfrak{A}$.

We call $\tilde{\omega}$ the *variation* of $f(x)$ in $\mathfrak{A}$. If $\tilde{\omega}$ is infinite, we say $f(x)$ has *unlimited variation* in $\mathfrak{A}$. If $f(x)$ is unlimited in $\mathfrak{A}$, it cannot be a function with limited variation in $\mathfrak{A}$.

510. Let $D = (a_1 a_2 \cdots)$ be a division of $\mathfrak{A}$. Let us form a new division Δ by interpolating a point α between $a_{\iota-1}$, a_ι.

Then δ_ι falls into two intervals δ'_ι, δ''_ι in Δ.



Let $M_\iota,\ M'_\iota,\ M''_\iota$

be the maximum of f in $\delta_\iota,\ \delta'_\iota,\ \delta''_\iota$,

and let $m_\iota,\ m'_\iota,\ m''_\iota$,

be the minimum of f in these intervals.

Then the term

$$\omega_\iota = M_\iota - m_\iota$$

in ω_D is replaced by

$$\omega'_\iota + \omega''_\iota = (M'_\iota - m'_\iota) + (M''_\iota - m''_\iota)$$

in ω_Δ. Now at least one of the M'_ι, M''_ι equals M_ι; and at least one of the m''_ι, m'_ι equals m_ι. To fix the ideas, let

$$M'_\iota = M_\iota, \quad m''_\iota = m_\iota.$$

Then

$$\omega_\Delta - \omega_D = (\omega'_\iota + \omega''_\iota) - \omega_\iota = M''_\iota - m'_\iota \geqq 0. \qquad (1$$

511. 1. *Let $f(x)$ be a limited monotone function in $\mathfrak{A}$. It has limited variation in $\mathfrak{A}$.*

To fix the ideas, let it be monotone increasing. Then

$$\omega_D = \{f(a_1) - f(a)\} + \{f(a_2) - f(a_1)\} + \cdots + \{f(b) - f(a_{n-1})\}$$
$$= f(b) - f(a).$$

Thus, whatever division D is employed, ω_D has the same value. Hence

$$\tilde{\omega} = f(b) - f(a).$$

2. *Let* $\mathfrak{A} = \mathfrak{A}_1 + \mathfrak{A}_2 \cdots + \mathfrak{A}_m$.

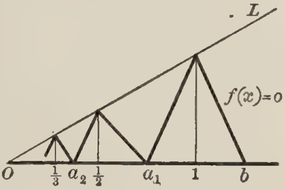
Let $f(x)$ be monotone and limited in each interval $\mathfrak{A}_\kappa$. Then f has limited variation in $\mathfrak{A}$.

For, we get the maximum value of ω_D when D embraces all the end points of the intervals $\mathfrak{A}_\kappa$. In fact, let D be a division which does not include one of these end points, say α, which lies in the interval δ_ι. Let Δ be a division formed by adding α to D. Then, by 510, 1),

$$\omega_\Delta \geqq \omega_D.$$

If, on the other hand, D is a division including all the end points of $\mathfrak{A}_1, \mathfrak{A}_2, \cdots$ we cannot increase ω_D by adding other points to D, as we saw in 1. Thus the variation of $f(x)$ in $\mathfrak{A}$ is the sum of the variations in each $\mathfrak{A}_\kappa$. As these latter are finite by 1, f is of limited variation in $\mathfrak{A}$.

512. 1. It is easy to construct functions having an infinite number of oscillations in $\mathfrak{A}$, which are of limited or unlimited variation.



Let $b > 1$, and in $\mathfrak{A} = (0, b)$ take the aggregate

$$1, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4}, \cdots$$

Let the line OL make the angle 45° with the x-axis. Between each pair of points

$$\frac{1}{m}, \qquad \frac{1}{m+1}$$

take a point a_m.

Let $f(x)$ have the graph formed of the heavy lines in the figure.

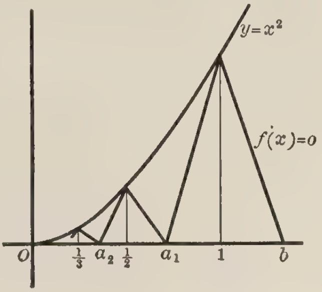
Let

$$D = \left(\frac{1}{n}, \ a_{n-1}, \ \frac{1}{n-1}, \ \cdots \ a_1, \ 1\right).$$

Then

$$\omega_D = 2\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right).$$

As we shall see later, the limit of the expression on the right is infinite.



Hence $f(x)$ is of unlimited variation.

2. To form a function having limited variation, take the parabola,

$$y = x^2,$$

instead of the right line OL in the last example.

Then

$$\omega_D = 2\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2}\right).$$

But as we shall see, the limit of the right side is here finite. Hence $f(x)$ is of limited variation.

3. Similar considerations show that

$$y = x \sin \frac{1}{x},\ x \neq 0; \qquad y = 0 \text{ for } x = 0$$

has unlimited variation in $(0, b)$; while

$$y = x^2 \sin \frac{1}{x},\ x \neq 0; \qquad y = 0 \text{ for } x = 0$$

has limited variation in $(0, b)$.

513. *If $f(x)$ has limited variation in $\mathfrak{A}$, it is integrable in $\mathfrak{A}$.*

We apply the criterion of 498. Let then, ω, σ be an arbitrary pair of positive numbers. Let D be a division of norm δ. Let ν be the number of subintervals in which the oscillation of $f(x)$ is $> \omega$. Then, for any division whatever,

$$\nu\omega \leq \tilde{\omega},$$

where $\tilde{\omega}$ is the total variation of f in $\mathfrak{A}$.

Let

$$\rho = \frac{\tilde{\omega}}{\omega};$$

then

$$\nu \leq \rho.$$

Let us take $\delta < \frac{\sigma}{\rho}$. Then the sum of the intervals in which the oscillation of f is $\geqq \omega$ is

$$\leq \nu\delta \leq \rho\delta < \sigma.$$

Content of Point Aggregates

514. 1. Let Δ be a point aggregate lying in the interval $\mathfrak{A}$. For example, Δ may be the interval $\mathfrak{A}$ itself. Or it may consist of a certain number of partial intervals of $\mathfrak{A}$. Or it may embrace an infinity of subintervals with or without their end points after the manner of Ex. 7, 8 in 271. Or it may consist of a mixed system of intervals and points not forming intervals.

Let us effect a division D of $\mathfrak{A}$ into subintervals

$$\delta_1, \ \delta_2, \ \cdots \ \delta_n$$

as heretofore.

Let

$$\delta'_1, \ \delta'_2, \ \cdots \tag{1}$$

be those intervals of D in which points of Δ fall. Let

$$\delta''_1, \ \delta''_2, \ \cdots \tag{2}$$

be those intervals of D, all of whose points lie in Δ.

Then

$$\overline{\Delta} = \lim_{\delta=0} \Sigma\delta'_\kappa, \qquad \underline{\Delta} = \lim_{\delta=0} \Sigma\delta''_\kappa$$

exist and are finite.

In fact, let us introduce an auxiliary function $f(x)$ whose value is 0 in $\mathfrak{A}$, except at the points of Δ, where its value is 1.

Then evidently, using the notation and results of 490, 491,

$$\sum_D \delta_\kappa = \Sigma M_\kappa \delta_\kappa = \overline{S}_D,$$

since $M_\kappa = 1$ if δ_κ is in 1), but is otherwise 0.

$$\sum_D \delta''_\kappa = \Sigma m_\kappa \delta_\kappa = \underline{S}_D,$$

since $m_\kappa = 1$ if δ_κ is in 2), but is otherwise 0.

Hence

$$\overline{\Delta} = \overline{S}, \qquad \underline{\Delta} = \underline{S}.$$

2. The numbers $\overline{\Delta}$, $\underline{\Delta}$ are called the *upper* and *lower content* of Δ.

When

$$\overline{\Delta} = \underline{\Delta},$$

their common value is called the *content* of Δ. We denote it by

$$\mathrm{Cont}\, \Delta,$$

or when no confusion can arise, by Δ.

The upper and lower contents may be denoted by

$$\overline{\Delta} = \overline{\mathrm{Cont}}\, \Delta, \qquad \underline{\Delta} = \underline{\mathrm{Cont}}\, \Delta.$$

A limited point aggregate having content is said to be *measurable*.

515. *Let $\mathfrak{B}$ be a partial aggregate of $\mathfrak{A}$. Let $\mathfrak{C} = \mathfrak{A} - \mathfrak{B}$ be the complementary aggregate. If $\mathfrak{A}$ and $\mathfrak{B}$ are measurable, so is $\mathfrak{C}$, and*

$$\text{Cont}\,\mathfrak{C} = \text{Cont}\,\mathfrak{A} - \text{Cont}\,\mathfrak{B}. \tag{1}$$

For, let D be a division of norm δ. Let $\mathfrak{F}$ be the frontier of $\mathfrak{B}$. Let $\overline{\mathfrak{A}}_D$, $\overline{\mathfrak{B}}_D$, $\overline{\mathfrak{C}}_D$, $\overline{\mathfrak{F}}_D$ be the sum of the intervals of D containing points of $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{F}$, respectively. Let $\underline{\mathfrak{A}}_D$, $\underline{\mathfrak{B}}_D$, $\underline{\mathfrak{C}}_D$ be the sum of the intervals which lie wholly in $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, respectively.

Then

$$\overline{\mathfrak{A}}_D \leq \overline{\mathfrak{B}}_D + \overline{\mathfrak{C}}_D \leq \overline{\mathfrak{A}}_D + \overline{\mathfrak{F}}_D, \tag{2}$$

since some of the intervals of D may contain both points of $\mathfrak{B}$ and $\mathfrak{C}$, and are therefore counted twice on the middle member of 2). Similarly,

$$\underline{\mathfrak{A}}_D \geqq \underline{\mathfrak{B}}_D + \underline{\mathfrak{C}}_D \geqq \underline{\mathfrak{A}}_D - \overline{\mathfrak{F}}_D. \tag{3}$$

Passing to the limit, $\delta = 0$, in 2), 3), we get

$$\overline{\mathfrak{A}} \leq \overline{\mathfrak{B}} + \overline{\mathfrak{C}} \leq \overline{\mathfrak{A}},$$

$$\underline{\mathfrak{A}} > \underline{\mathfrak{B}} + \underline{\mathfrak{C}} \geq \underline{\mathfrak{A}}.$$

But $\qquad \overline{\mathfrak{A}} = \underline{\mathfrak{A}} = \text{Cont}\,\mathfrak{A}, \qquad \overline{\mathfrak{B}} = \underline{\mathfrak{B}} = \text{Cont}\,\mathfrak{B}.$

Hence

$$\overline{\mathfrak{C}} = \underline{\mathfrak{C}} = \text{Cont}\,\mathfrak{A} - \text{Cont}\,\mathfrak{B},$$

which gives 1).

516. 1. By the aid of the auxiliary function $f(x)$ introduced in 514, 1, the criteria for integrability which we deduced in 495, 497 give at once criteria that Δ have a content. Thus 495 gives:

In order that Δ have content, it is necessary and sufficient that the sum of those intervals containing both points of Δ and points not in Δ, converge to zero, as the norm δ of D converges to zero.

2. From 497 we have:

In order that Δ have content, it is necessary and sufficient that for each positive number ϵ there exists a division D of $\mathfrak{A}$, such that the sum of the intervals in which both points of Δ and not of Δ occur, is $< \epsilon$.

EXAMPLES

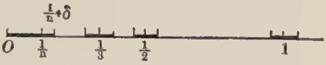
517. 1. $\Delta =$ rational numbers in $\mathfrak{A} = (a, b)$.

Here

$$\overline{\Delta} = b - a, \quad \underline{\Delta} = 0.$$

2.

$$\Delta = 0,\ 1,\ \tfrac{1}{2},\ \tfrac{1}{3},\ \tfrac{1}{4},\ \cdots$$



Let $\epsilon > 0$ be arbitrarily small.

Let us define the division D as follows. Inclose each of the points

$$1,\ \frac{1}{2},\ \frac{1}{3},\ \cdots\ \frac{1}{n-1}$$

within intervals of length

$$\frac{\epsilon}{3n},$$

where

$$n > \frac{3}{\epsilon}.$$

The remaining points of Δ fall in the interval

$$\left(0,\ \frac{1}{n} + \delta\right), \qquad \delta = \frac{\epsilon}{3}.$$

Then the sum of the intervals containing both points in Δ and not in Δ is $< \epsilon$. Hence, by 516, 2, Δ has a content.

As obviously

$$\underline{\Delta} = 0,$$

we have

$$\text{Cont}\ \Delta = 0.$$

If E is the complement of Δ in $\mathfrak{A}$,

$$\text{Cont}\ E = \text{Cont}\ \mathfrak{A} = (b - a),$$

by 515.

518. 1. A point aggregate of content zero is called *discrete*. Such an aggregate is given in 517, Ex. 2.

Every limited point aggregate of the first species is discrete.

The reasoning is perfectly analogous to that of 501.

2. *Let $f(x)$ be limited in the interval $\mathfrak{A}$. If the points of discontinuity of $f(x)$ form a discrete aggregate, $f(x)$ is integrable in $\mathfrak{A}$.*

This follows at once from 497.

3. *Let $y = f(x)$ be univariant in $\mathfrak{A}$. Let*

$$\left|\frac{\Delta y}{\Delta x}\right| \leq M, \quad \textit{if}\ \Delta x \leq d.$$

Let Δ *be a discrete aggregate in* $\mathfrak{A}$. *The image* E *of* Δ *is also discrete.*

Let us effect a division of $\mathfrak{A}$ of norm $\delta < d$.

Let $\delta_1, \delta_2, \cdots$ be the subintervals containing points of Δ. Let $\eta_1, \eta_2, \cdots$ be the corresponding intervals on the y-axis. Then, by hypothesis,

$$\eta_\kappa \leq M\delta_\kappa.$$

Thus

$$\Sigma\eta_\kappa \leq M\Sigma\delta_\kappa. \tag{1}$$

But Δ being discrete, we can take δ so small that the right side of 1) is $< \epsilon$.

Hence E is discrete.

Generalized Definition of an Integral

519. Up to the present we have supposed that the integrand $f(x)$ is defined for all the points of the interval $\mathfrak{A} = (a, b)$. By employing the results of the last articles, we can generalize as follows:

Let $\mathfrak{B}$ be a measurable aggregate in $\mathfrak{A}$, and let $f(x)$ be a *limited* function defined over $\mathfrak{B}$. Let us effect a division

$$D(a_1, a_2, \cdots)$$

of $\mathfrak{A}$ of norm δ.

Those intervals

$$(a_{\iota_1}, a_{\iota_1+1}), \quad (a_{\iota_2}, a_{\iota_2+1}) \cdots \tag{1}$$

all of whose points lie in $\mathfrak{B}$, form a system which we denote by D_1.

The lengths of the intervals 1) we denote by $\delta_1', \delta_2', \cdots$; while $\xi_1', \xi_2', \cdots$ are points taken at pleasure, one in each interval of 1).

Let us build the sum

$$J_\delta = \sum_{D_1} f(\xi_\kappa')\delta_\kappa'.$$

If as $\delta \doteq 0$, J_δ converges to one and the same value, however the divisions D and the ξ's be chosen, we call this common limit the integral of $f(x)$ over $\mathfrak{B}$, and denote it by

$$\int_{\mathfrak{B}} f(x)dx.$$

We say in this case that $f(x)$ is integrable with respect to $\mathfrak{B}$.

520. *Let $f(x)$ be a limited function defined over a limited point aggregate $\mathfrak{B}$ of content zero. Then $f(x)$ is integrable over $\mathfrak{B}$, and*

$$\int_{\mathfrak{B}} f(x)dx = 0.$$

Let

$$J_\delta = \sum_{D_1} f(\xi'_\kappa)\delta'_\kappa$$

have the same meaning as in 519.

Since f is limited, let

$$|f(x)| < M.$$

Then

$$|J_\delta| < M\Sigma\delta'_\kappa.$$

Since $\mathfrak{B}$ is of content zero,

$$\lim_{\delta=0} \Sigma\delta'_\kappa = 0.$$

Hence

$$\lim_{\delta=0} J_\delta = 0,$$

which proves the theorem.

521. 1. *Let $f(x)$ be a limited integrable function defined over the measurable aggregate $\mathfrak{B}$. Let the interval $\mathfrak{A} = (a, b)$ contain $\mathfrak{B}$.*

Let

$$g(x) = f(x), \qquad \textit{in } \mathfrak{B};$$

$$= 0, \qquad \textit{for points of } \mathfrak{A} \textit{ not in } \mathfrak{B}.$$

Then $g(x)$ is integrable in (a, b), and

$$\int_a^b g(x)dx = \int_{\mathfrak{B}} f(x)dx. \tag{1}$$

Let us effect the division D of $\mathfrak{A}$ as in 519.

As before, let D_1 be the system of intervals lying in $\mathfrak{B}$. Let D_2 be the system of intervals containing no point of $\mathfrak{B}$. Let Δ be the system of intervals containing both points of $\mathfrak{B}$ and points not in $\mathfrak{B}$.

We build now the sum

$$J_\delta = \sum_D g(\xi_\kappa)\delta_\kappa$$

with reference to the interval $\mathfrak{A}$.

Since

$$D = D_1 + D_2 + \Delta,$$

we have

$$J_\delta = \sum_{D_1} g(\xi_\kappa)\delta_\kappa + \sum_{D_2} g(\xi_\kappa)\delta_\kappa + \sum_{\Delta} g(\xi_\kappa)\delta_\kappa. \qquad (2$$

Since $g(x) = 0$ in D_2, the second sum in 2) is 0. Since

$$f(x) = g(x)$$

in D_1, we have now

$$J_\delta = \sum_{D_1} f(\xi_\kappa)\delta_\kappa + \sum_{\Delta} g(\xi_\kappa)\delta_\kappa. \qquad (3$$

Now, by hypothesis, f is integrable in $\mathfrak{B}$, and

$$\lim_{\delta=0} \sum_{D_1} f(\xi_\kappa)\delta_\kappa = \int_{\mathfrak{B}} f dx.$$

On the other hand,

$$\lim_{\delta=0} \sum_{\Delta} g(\xi_\kappa)\delta_\kappa = 0,$$

by 516 and 520.

Hence, passing to the limit, $\delta = 0$, in 3), we have 1).

2. The reasoning in 1 gives as corollary:

If $f(x)$ is limited in $\mathfrak{B}$, and $g(x)$ is integrable in $\mathfrak{A}$, then $f(x)$ is integrable in $\mathfrak{B}$, and

$$\int_a^b g(x)dx = \int_{\mathfrak{B}} f(x)dx.$$

This is at once evident, on passing to the limit in 3).

3. *Let $f(x)$ be limited in $\mathfrak{A} = (a, b)$. Let Δ be a discrete aggregate in $\mathfrak{A}$. Let $\mathfrak{B} = \mathfrak{A} - \Delta$. Let $f(x)$ be integrable in $\mathfrak{B}$. Then $f(x)$ is integrable in $\mathfrak{A}$, and*

$$\int_a^b f dx = \int_{\mathfrak{B}} f dx.$$

The demonstration is similar to 1, omitting the system D_2.

522. 1. *Let $f(x)$ be a limited integrable function with respect to the measurable aggregate $\mathfrak{B}$, lying in the interval $\mathfrak{A} = (a, b)$. Let $D = (a_1, a_2, \cdots a_{n-1})$ be a division of $\mathfrak{A}$ of norm δ. Let*

$$(\alpha_1\beta_1),\ (\alpha_2\beta_2),\ \cdots$$

be resulting intervals formed of one or several contiguous intervals of D, lying in $\mathfrak{B}$. Then

$$\int_{\mathfrak{B}} f dx = \lim_{\delta=0} \sum_{\kappa} \int_{\alpha_\kappa}^{\beta_\kappa} f dx. \tag{1}$$

Let us introduce the auxiliary function $g(x)$ of 521. Then, by 521,

$$\int_{\mathfrak{B}} f dx = \int_a^b g dx.$$

Now the division D breaks $\mathfrak{A}$ up into the intervals

$$(a, a_1), \ (a_1, a_2), \ (a_2, a_3), \ \cdots \ (a_{n-1}, b).$$

Letting D_1, D_2, Δ have the same meaning as in 521, we have

$$\int_a^b g dx = \sum_D \int_{a_\iota}^{a_{\iota+1}} g dx = \sum_{D_1} \int_{a_\iota}^{a_{\iota+1}} g dx + \sum_{D_2} \int_{a_\iota}^{a_{\iota+1}} g dx + \sum_{\Delta} \int_{a_\iota}^{a_{\iota+1}} g dx.$$

But

$$\sum_{D_1} \int_{a_\iota}^{a_{\iota+1}} g dx = \sum_{D_1} \int_{a_\iota}^{a_{\iota+1}} f dx = \sum \int_{\alpha_\kappa}^{\beta_\kappa} f dx ;$$

while

$$\sum_{D_2} \int_{a_\iota}^{a_{\iota+1}} g dx = 0,$$

since $g = 0$ in the intervals of D_2.

Thus,

$$\int_a^b g dx = \sum \int_{\alpha_\kappa}^{\beta_\kappa} f dx + \sum_{\Delta} \int_{a_\iota}^{a_{\iota+1}} g dx. \tag{2}$$

Now, if

$$|g| \leq M,$$

we have, by 489, 4,

$$\left| \sum_{\Delta} \int_{a_\iota}^{a_{\iota+1}} g dx \right| \leq M \sum_{\Delta} (a_{\iota+1} - a_\iota) = M\Delta.$$

But since $\mathfrak{B}$ is measurable,

$$\lim_{\delta=0} \Delta = 0.$$

Hence the second term on the right of 2) has the limit 0. Hence, passing to the limit in 2), we get 1).

2. The preceding reasoning gives the corollary:

If $f(x)$ *is limited in* $\mathfrak{B}$ *and integrable in each of the intervals* $(\alpha_\kappa, \beta_\kappa)$, *and* $\sum\int_{\alpha_\kappa}^{\beta_\kappa} fdx$ *is convergent as* $\delta \doteq 0$, *then*

$$\int_{\mathfrak{B}} fdx = \lim_{\delta=0} \sum \int_{\alpha_\kappa}^{\beta_\kappa} fdx.$$

This is evident on passing to the limit in 2).

3. *Let* $f(x)$ *be limited in* $\mathfrak{A} = (a, b)$. *Let* Δ *be a discrete aggregate in* $\mathfrak{A}$. *Let* $\mathfrak{B} = \mathfrak{A} - \Delta$. *Then*

$$\int_a^b fdx = \lim_{\delta=0} \sum \int_{\alpha_\kappa}^{\beta_\kappa} fdx,$$

provided the limit on the right is finite.

For, by 2,

$$\lim_{\delta=0} \sum \int_{\alpha_\kappa}^{\beta_\kappa} fdx = \int_{\mathfrak{B}} fdx.$$

But, by 521, 3,

$$\int_{\mathfrak{B}} fdx = \int_a^b fdx.$$

CHAPTER XIII

PROPER INTEGRALS

First Properties

523. In the last chapter the integrand $f(x)$, as well as the interval of integration $\mathfrak{A}$, were limited. Integrals for which this is the case are called *proper integrals*, in contradistinction to those in which either $f(x)$ or $\mathfrak{A}$ is unlimited. These latter are called *improper integrals.*

In this chapter we consider *only* proper integrals. We wish to establish their more elementary properties.

In $\mathfrak{A} = (a, b)$, we shall take $a < b$, unless the contrary is stated. All the functions employed as integrands are supposed to be limited and integrable in $\mathfrak{A}$.

524. For the sake of completeness, we begin by stating the three following properties already established respectively in 489, 2; 489, 4; 504, viz.:

$$\int_a^b f(x)dx = -\int_b^a f(x)dx, \qquad a \lesseqgtr b. \qquad (1$$

$$\left|\int_a^b f(x)dx\right| \le M|b-a|, \qquad a \lesseqgtr b, \qquad (2$$

$|f(x)|$ being $\le M$ in the interval (a, b).

$$\int_a^b \{c_1 f_1 + \cdots + c_s f_s\} dx = c_1 \int_a^b f_1 dx + \cdots + c_s \int_a^b f_s dx, \quad a \gtreqless b. \quad (3$$

525. *Let a, b, c, be three points in any order. Then*

$$\int_a^b f dx = \int_a^c f dx + \int_c^b f dx. \qquad (1$$

Suppose first that $a < c < b$. Since

$$\lim_{\delta=0} \Sigma f(\xi_\kappa)\delta_\kappa$$

is the same, whatever system of division D we choose, let us consider only such divisions in which c enters. The points of D which fall in (a, c), let us call D_1; those falling in (c, b), call D_2.

Then

$$\sum_D f\delta_\kappa = \sum_{D_1} f\delta_\kappa + \sum_{D_2} f\delta_\kappa. \qquad (2$$

Now $f(x)$ being integrable in $\mathfrak{A}$, is integrable in (a, c) and (c, b), by 496.

Hence, passing to the limit in 2), we get 1).

The theorem is now readily established for any other order of a, b, c.

526. 1. *In* $\mathfrak{A} = (a, b)$, *let*

$$m \leq f(x) \leq M.$$

Then

$$m(b-a) \leq \int_{\mathfrak{A}} f dx \leq M(b-a). \qquad (1$$

For,

$$m \leq f(\xi_\kappa) \leq M.$$

Hence

$$\Sigma m\delta_\kappa \leq \Sigma f(\xi_\kappa)\delta_\kappa \leq \Sigma M\delta_\kappa,$$

or

$$m(b-a) \leq J_\delta \leq M(b-a).$$

Passing to the limit, $\delta = 0$, we get 1).

2. *In* $\mathfrak{A}$ *let* $f(x) \geqq g(x)$. *Then*

$$\int_{\mathfrak{A}} f dx \geq \int_{\mathfrak{A}} g dx. \qquad (2$$

For,

$$h(x) = f(x) - g(x) \geqq 0, \qquad \text{in } \mathfrak{A}.$$

Hence, by 1),

$$\int_{\mathfrak{A}} h dx = \int_{\mathfrak{A}} f dx - \int_{\mathfrak{A}} g dx \geqq 0,$$

which gives 2).

527. 1. We saw in 508 that, if $f(x)$ is integrable in $\mathfrak{A}$, it must have points of continuity c, in any subinterval of $\mathfrak{A}$. This fact leads us to state the following theorem:

Let $f(x) \geqq 0$ *in* $\mathfrak{A}$. *If* f *is continuous at* c, *and* $f(c) > 0$, *then*

$$\int_{\mathfrak{A}} f dx > 0.$$

To fix the ideas, suppose c is an inner point. Then by 351, 2, there exists an interval (c', c'') about c, in which $f(x) > \rho > 0$.

Hence by 526,

$$\int_a^{c'} f dx \geqq 0, \quad \int_{c'}^{c''} f dx \geqq \rho(c'' - c') = \sigma, \quad \int_{c''}^{b} f dx \geqq 0.$$

But

$$\int_{\mathfrak{A}} f dx = \int_a^{c'} + \int_{c'}^{c''} + \int_{c''}^{b} \geqq \sigma > 0.$$

2. As corollary we have:

Let $f(x) \geqq g(x)$ *in* $\mathfrak{A}$. *If at a point* c *of continuity of* f *and* g, $f(c) > g(c)$, *then*

$$\int_{\mathfrak{A}} f dx > \int_{\mathfrak{A}} g dx.$$

3. By means of the preceding inequalities, we can often estimate approximately the value of an integral with little labor, as the following examples show.

Ex. 1.
$$.5 < \int_0^{\frac{1}{2}} \frac{dx}{\sqrt{1 - x^n}} < .5236, \qquad n > 2. \tag{1}$$

For, if $0 < x < 1$,

$$1 < \frac{1}{\sqrt{1 - x^n}} < \frac{1}{\sqrt{1 - x^2}}.$$

Hence *

$$\int_0^{\frac{1}{2}} dx < \int^{\frac{1}{2}} \frac{dx}{\sqrt{1 - x^n}} < \int_0^{\frac{1}{2}} \frac{dx}{\sqrt{1 - x^2}} = \text{arc sin } \tfrac{1}{2} = \frac{\pi}{6},$$

which gives 1).

Ex. 2.
$$x e^{-x^2} < \int_0^x e^{-u^2} du < \text{arctg } x, \qquad x > 0. \tag{2}$$

For by 413, 2,

$$e^z = 1 + z + \frac{z^2}{2} e^{\theta z}, \qquad 0 < \theta < 1.$$

---

* In order to illustrate these and a few immediately following theorems we assume the elementary properties of indefinite integrals, which are treated in 536 *seq*.

Hence $$e^z > 1 + z, \qquad z > 0.$$

Thus $$e^{-x^2} < e^{-u^2} < \frac{1}{1+u^2}, \qquad \text{if } 0 < u < x.$$

Hence, x being a constant,

$$\int_0^x e^{-x^2} du < \int_0^x e^{-u^2} du < \int_0^x \frac{du}{1+u^2},$$

which gives 2).

528. 1. *Let $f(x)$ be limited and integrable in $\mathfrak{A}$; then $|f(x)|$ is integrable in $\mathfrak{A}$, and*

$$\left|\int_{\mathfrak{A}} f dx\right| \leq \int_{\mathfrak{A}} |f| dx. \tag{1}$$

In the first place, $|f|$ is integrable by 507.

On the other hand,

$$-|f(x)| \leq f(x) \leq |f(x)|.$$

The relation 1) follows now by 526.

2. The reader should guard against the error of assuming that $f(x)$ is necessarily integrable in $\mathfrak{A}$ if $|f(x)|$ is integrable in $\mathfrak{A}$.

For example, let

$$f(x) = 1 \text{ for rational } x \text{ in } \mathfrak{A},$$
$$= -1 \text{ for irrational } x \text{ in } \mathfrak{A}.$$

Then $$|f(x)| = 1, \qquad \text{for every } x \text{ in } \mathfrak{A},$$

and is therefore integrable in $\mathfrak{A}$. But $f(x)$ is obviously not integrable in $\mathfrak{A}$.

529. 1. *In $\mathfrak{A} = (a, b)$, let*

$$m \leq g(x) \leq M. \tag{1}$$

When not zero, let $f(x)$ be positive. Then

$$m \int_{\mathfrak{A}} f dx \leq \int_{\mathfrak{A}} fg dx \leq M \int_{\mathfrak{A}} f dx. \tag{2}$$

For, multiplying 1) by $f(x)$, we have

$$mf \leq fg \leq Mf.$$

We have now 2) by 526.

2. *In* $\mathfrak{A}$ *let* $f(x) \geqq 0$. *Let* $m \leq g(x) \leq M$. *At a point* c *of continuity of* $f(x)$, $g(x)$, *let*

$$f(c) > 0, \qquad m < g(c) < M.$$

Then

$$m\int_{\mathfrak{A}} f dx < \int_{\mathfrak{A}} fg dx < M\int_{\mathfrak{A}} f dx.$$

We have only to apply 527, 1 to the functions

$$(M-g)f \text{ and } (g-m)f.$$

Ex. 1.

$$\arcsin x < \int_0^x \frac{dx}{\sqrt{1-x^2\cdot 1-\lambda x^2}} < \frac{\arcsin x}{\sqrt{1-\lambda}}, \tag{3}$$

where

$$0 < \lambda < 1, \quad 0 < x < 1.$$

For,

$$1 < \frac{1}{\sqrt{1-\lambda x^2}} < \frac{1}{\sqrt{1-\lambda}}.$$

Hence

$$\int_0^x \frac{dx}{\sqrt{1-x^2}} < \int_0^x \frac{dx}{\sqrt{1-x^2}\sqrt{1-\lambda x^2}} < \frac{1}{\sqrt{1-\lambda}}\int_0^x \frac{dx}{\sqrt{1-x^2}},$$

which gives 3).

Ex. 2.

$$.333 < \int_0^1 x^2 e^{x^2} dx < .907. \tag{4}$$

For, if $0 < x < 1$,

$$1 < e^{x^2} < e.$$

Hence

$$\int_0^1 x^2 dx < \int_0^1 x^2 e^{x^2} dx < e\int_0^1 x^2 dx,$$

which gives 4).

530 1. *Let* $f(x) = g(x)$ *in* $\mathfrak{A}$, *except for the points of a discrete aggregate* Δ; *then* f, g *being limited and integrable,*

$$\int_{\mathfrak{A}} f dx = \int_{\mathfrak{A}} g dx. \tag{1}$$

For, set

$$h(x) = f(x) - g(x).$$

Then $h = 0$, except for the points of Δ.

Let D be a division of $\mathfrak{A}$. Let D_1 embrace those intervals containing no points of Δ. Let D_2 embrace the other intervals of D. Then

$$\sum_D h(\xi\ )\delta_\kappa = \sum_{D_1} + \sum_{D_2} = \sum_{D_2}.$$

Let now $\delta \doteq 0$. The limit of the left side is

$$\int_{\mathfrak{A}} h dx.$$

Since Δ is discrete, the limit of the right side is 0 by 520. Hence

$$\int_{\mathfrak{A}} h dx = 0,$$

which proves 1).

2. As a corollary we have:

In the integral

$$J = \int_{\mathfrak{A}} f dx$$

we may change at pleasure the value of $f(x)$ at the points of an arbitrary discrete aggregate, without changing the value of J, provided the new integrand is also limited in $\mathfrak{A}$.

First Theorem of the Mean

531. *Let $f(x)$, $g(x)$ be limited and integrable in $\mathfrak{A} = (a, b)$. When not 0, let $f(x)$ be positive.*

Then

$$\int_a^b fg dx = G \int_a^b f dx, \qquad a \lesseqgtr b, \tag{1}$$

where $G = \text{Mean } g(x)$, *in* $\mathfrak{A}$.

For, by definition, 268, 4,

$$m \leq G \leq M,$$

where m and M are respectively the minimum and maximum of $g(x)$ in $\mathfrak{A}$. Also, by 529,

$$m \int_{\mathfrak{A}} f dx \leq \int_{\mathfrak{A}} fg dx \leq M \int_{\mathfrak{A}} f dx, \qquad a < b,$$

which gives 1) in this case. The case of $a > b$ follows now at once.

The above is called the *first theorem of the mean.* We give now some special cases of it.

532. *Let $f(x)$ be integrable in $\mathfrak{A} = (a, b)$. Let*

$$\mathfrak{M} = \mathrm{Mean}\, f(x), \qquad in\ \mathfrak{A}.$$

Then

$$\int_a^b f(x)\,dx = \mathfrak{M}(b-a). \qquad a \lesseqgtr b.$$

Proved, by taking one of the functions in 531, equal to 1.

533. *In the interval $\mathfrak{A} = (a, b)$, let $f(x)$ be limited, integrable, and non negative. Let $g(x)$ be continuous. Then*

$$\int_{\mathfrak{A}} fg\,dx = g(\xi)\int_{\mathfrak{A}} f\,dx, \qquad a \le \xi \le b. \tag{1}$$

For, by 531, setting $G = \mathrm{Mean}\, g(x)$,

$$\int_{\mathfrak{A}} fg\,dx = G\int_{\mathfrak{A}} f\,dx.$$

But $g(x)$ being continuous, takes on every value between its extremes including end values, while x runs over $\mathfrak{A}$, by 357. Hence for some ξ in $\mathfrak{A}$,

$$g(\xi) = G,$$

which proves 1).

534. *In the interval $\mathfrak{A} = (a, b)$ let $f(x)$ be limited, integrable and non negative. Let $g(x)$ be continuous. At some point of continuity of $f(x)$ let*

$$m < g(x) < M,$$

where $m = \mathrm{Min}\, g(x)$, $M = \mathrm{Max}\, g(x)$, in $\mathfrak{A}$.

Then

$$\int_{\mathfrak{A}} fg\,dx = g(\xi)\int_{\mathfrak{A}} f\,dx. \qquad a < \xi < b.$$

For, by 527, 2,

$$m\int_{\mathfrak{A}} f\,dx < \int_{\mathfrak{A}} fg\,dx < M\int_{\mathfrak{A}} f\,dx.$$

Hence,

$$\int_{\mathfrak{A}} fg\,dx = G\int_{\mathfrak{A}} f\,dx, \qquad m < G < M.$$

Now $g(x)$ being continuous takes on its minimum m at some point α, and its maximum at some point β in $\mathfrak{A}$, by 357. Moreover, at some point ξ in the interval (α, β), $g(x)$ must take on the value G. As $g(x)$ has the values m and M at the end points of this interval, ξ must lie *within* this interval. Hence

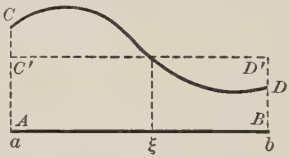
$$a < \xi < b.$$

535. *Let $f(x)$ be continuous in $\mathfrak{A} = (a, b)$. Then*

$$\int_{\mathfrak{A}} f dx = (b - a) f(\xi), \qquad a < \xi < b.$$

This is a corollary of 534, one of the functions being 1, provided $f(x)$ is not a constant, when the theorem is obviously true.

This theorem admits a simple geometric interpretation. It states that there is a point ξ, $a < \xi < b$, such that the area of the rectangle $ABC'D'$ is the same as that of the figure $ABCD$, determined by the graph of $f(x)$.



The Integral considered as a Function of its Upper Limit

536. *Let $f(x)$ be limited and integrable in $\mathfrak{A} = (a, b)$. Then*

$$F(x) = \int_a^x f(x) dx, \qquad a, x \text{ in } \mathfrak{A},$$

is a one-valued continuous function of x in $\mathfrak{A}$.

Since $f(x)$ is integrable in $\mathfrak{A}$, F has one, and only one, value for every x.

Let
$$|f(x)| \leq M.$$
We have
$$\Delta F = F(x + h) - F(x) = \int_a^{x+h} f(x) dx - \int_a^x f(x) dx = \int_x^{x+h} f(x) dx.$$

Hence
$$|\Delta F| \leq M |h| \qquad \text{by 524, 2).}$$

Thus if we take

$$\delta < \frac{\epsilon}{M},$$

we have

$$|\Delta F| < \epsilon,$$

for every h, such that $x + h$ falls in $\mathfrak{A}$, and

$$|h| < \delta.$$

Hence $F(x)$ is continuous in $\mathfrak{A}$.

537. 1. *Let $f(x)$ be limited and integrable in $\mathfrak{A} = (a, b)$. Let*

$$J(x) = \int_\alpha^x f(x)dx, \qquad \alpha,\ x \text{ in } \mathfrak{A}.$$

If f is continuous at x,

$$\frac{dJ}{dx} = f(x). \tag{1}$$

To fix the ideas, let

$$a \leq \alpha < x < x + h \leq b.$$

Then

$$\Delta J = J(x + h) - J(x) = \int_\alpha^{x+h} - \int_\alpha^x = \int_x^{x+h}$$
$$= h\mathfrak{M}, \quad \text{by 532,}$$

where $\mathfrak{M} = \text{Mean } f(x)$ in $(x, x + h)$.

But since $f(x)$ is continuous at x,

$$\mathfrak{M} = f(x) + \eta, \qquad |\eta| < \epsilon,$$

for any $h <$ some δ.

Hence

$$\frac{\Delta J}{\Delta x} = f(x) + \eta,$$

Passing to the limit, we get 1).

2. As a corollary we have:

Let $f(x)$ be limited and integrable in $\mathfrak{A} = (a, b)$. Then

$$\lim_{h=0} \frac{1}{h} \int_x^{x+h} f(x)dx = f(x), \qquad x \text{ in } \mathfrak{A}, \tag{2}$$

if f is continuous at the point x.

538. 1. *In the interval $\mathfrak{A}$, let $f(x)$ be a limited integrable function. Let $F(x)$ be a one-valued function whose derivative is $f(x)$. Then*

$$\int_\alpha^\beta f dx = F(\beta) - F(\alpha). \qquad \alpha, \beta \text{ in } \mathfrak{A}. \tag{1}$$

For, let $D = (a_1, a_2, \cdots a_{n-1})$ be a division of the interval (α, β). Then by the Law of the Mean,

$$\begin{aligned} F(a_1) - F(\alpha) &= f(\xi_1)\delta_1, \\ F(a_2) - F(a_1) &= f(\xi_2)\delta_2, \\ \cdot \quad \cdot \quad \cdot &\quad \cdot \quad \cdot \\ F(\beta) - F(a_{n-1}) &= f(\xi_n)\delta_n. \end{aligned} \tag{2}$$

Adding the equations 2), we get, since terms on the left cancel in pairs,

$$F(\beta) - F(\alpha) = \Sigma f(\xi_\kappa)\delta_\kappa. \tag{3}$$

Since $f(x)$ is integrable,

$$\lim_{\delta=0} \Sigma f(\xi_\kappa)\delta_\kappa = \int_\alpha^\beta f dx.$$

Passing therefore to the limit in 3), we get **1)**.

2. *Example.* By differentiation we verify

$$D_x \cdot \frac{x - \lambda \operatorname{arctg} (\lambda \tan x)}{1 - \lambda^2} = \frac{1}{1 + \lambda^2 \tan^2 x},$$

for $|\lambda| \neq 1$ and $x \neq (2m + 1)\pi/2$. Hence by 538, 1,

$$\int_0^x \frac{dx}{1 + \lambda^2 \tan^2 x} = \frac{x - \lambda \operatorname{arctg} (\lambda \tan x)}{1 - \lambda^2}, \qquad 0 < x < \frac{\pi}{2}, \qquad |\lambda| \neq 1.$$

The integrand is not defined for $x = \pi/2$; let us therefore assign it the value 0. The integral on the left is continuous, by 536. Hence

$$\int_0^{\pi/2} = L \lim_{x=\pi/2} \int_0^x = L \lim \frac{x - \lambda \operatorname{arc\,tg} (\lambda \tan x)}{1 - \lambda^2},$$

or

$$\int_0^{\pi/2} \frac{dx}{1 + \lambda^2 \tan^2 x} = \frac{\pi}{2} \frac{(1 \pm \lambda)}{1 - \lambda^2} = \frac{\pi}{2} \cdot \frac{1}{1 + |\lambda|}. \tag{4}$$

As we have derived this formula we have been obliged to assume $|\lambda| \neq 1$. It is, however, valid even when $|\lambda| = 1$. For,

$$\int_0^{\pi/2} \frac{dx}{1 + \tan^2 x} = \int_0^{\pi/2} \cos^2 x \, dx = \left[\frac{1}{2}x + \frac{1}{4} \sin 2x\right]_0^{\pi/2} = \frac{\pi}{4},$$

which agrees with 4) when $|\lambda| = 1$.

539. *Criticism.* A common form of demonstration of the preceding theorem is the following. Since

$$\lim_{h=0} \frac{F(x+h)-F(x)}{h} = f(x),$$

we have

$$\epsilon > 0, \qquad \delta > 0, \qquad \frac{F(x+h)-F(x)}{h} = f(x) + \epsilon', \qquad |\epsilon'| < \epsilon,$$

for $|h| < \delta$. The equations 2) of 538 are now written

$$F(a_1) - F(\alpha) = f(\alpha)h_1 + \epsilon_1 h_1,$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$F(\beta) - F(a_{n-1}) = f(a_{n-1})h_n + \epsilon_n h_n.$$

Adding, we get

$$F(\beta) - F(\alpha) = \Sigma f(a_\kappa)h_\kappa + \Sigma \epsilon_\kappa h_\kappa. \tag{1}$$

If ϵ is numerically the greatest of the ϵ_κ,

$$|\Sigma \epsilon_\kappa h_\kappa| \leq \epsilon \Sigma h_\kappa = \epsilon(\beta - \alpha), \qquad \beta > \alpha.$$

It is now assumed that $\epsilon \doteq 0$ with δ. Hence passing to the limit, $\delta = 0$, 1) gives 538, 1).

The objection to this demonstration lies in the tacit assumption that the difference quotient converges uniformly to the derivative. Cf. 404. In other words, that it is possible to divide the interval (α, β) into subintervals $h_1, h_2, \cdots h_n$ such that $\epsilon_1, \epsilon_2, \cdots \epsilon_n$ are all $\leq \sigma$, a positive number, small at pleasure. As elementary textbooks say nothing of uniform convergence, the above reasoning is incomplete.

Change of Variable

540. 1. *Let $f(x)$ be limited and integrable in $\mathfrak{A} = (a, b)$, $a \gtrless b$. Let*

$$u = \phi(x) \tag{1}$$

be a univariant function in $\mathfrak{A}$ having a continuous derivative $\phi'(x) \neq 0$. Let $\mathfrak{B} = (\alpha, \beta)$ be the image of $\mathfrak{A}$ afforded by 1). Let

$$x = \psi(u)$$

be the inverse function of ϕ. *Then, if* $f[\psi(u)]\psi'(u)$ *is integrable in* $\mathfrak{B}$,

$$\int_a^b f(x)dx = \int_\alpha^\beta f[\psi(u)]\psi'(u)du. \tag{2}$$

By 358, the correspondence between the two intervals $\mathfrak{A}$, $\mathfrak{B}$ is uniform.

Let $E(u_1, u_2, \cdots)$ be a division of $\mathfrak{B}$, of norm δ.

Let Δx_κ in $\mathfrak{A}$ correspond to Δu_κ in $\mathfrak{B}$.

By the Law of the Mean,

$$\Delta x_\kappa = \psi'(\eta_\kappa)\Delta u_\kappa, \qquad \eta_\kappa \textit{ lying in } \Delta u_\kappa.$$

Let ξ_κ in $\mathfrak{A}$ correspond to η_κ in $\mathfrak{B}$. Then

$$\Sigma f(\xi_\kappa)\Delta x_\kappa = \Sigma f[\psi(\eta_\kappa)]\psi'(\eta_\kappa)\Delta u_\kappa. \tag{3}$$

Since

$$f(x), \qquad f[\psi(u)]\psi'(u)$$

are limited, and integrable by hypothesis, we have 2) by passing to the limit in 3).

2. If the conditions of 1 are not satisfied in the intervals $\mathfrak{A}$, $\mathfrak{B}$, it may be possible to divide them up into subintervals, in each of which these conditions hold.

541. 1. Let us evaluate

$$J = \int_0^1 \frac{\log(1+x)dx}{1+x^2}. \tag{1}$$

We set

$$x = \tan u = \psi(u), \quad \text{or} \quad u = \text{arc} \tan x = \phi(x).$$

Then

$$\mathfrak{A} = (0, 1), \quad \mathfrak{B} = (0, \pi/4).$$

The conditions of 540 are obviously satisfied. Hence

$$J = \int_0^{\pi/4} \log(1 + \tan u)du.$$

Let us make a new transformation

$$u = \pi/4 - v.$$

The conditions of 540 being again satisfied,

$$J = \int_0^{\pi/4} \log\left\{1 + \tan\left(\frac{\pi}{4} - v\right)\right\} dv.$$

But

$$\tan\left(\frac{\pi}{4} - v\right) = \frac{1 - \tan v}{1 + \tan v}.$$

Hence

$$J = \int_0^{\pi/4} \log \frac{2\, dv}{1 + \tan v} = \log 2 \int_0^{\pi/4} dv - J,$$

or

$$2J = \log 2 \int_0^{\pi/4} dv = \frac{\pi}{4} \log 2.$$

Thus

$$J = \pi/8 \log 2. \qquad (2$$

2. That we should not affect a change of variable in a definite integral, without due precaution, is illustrated by the following example.

Let

$$\int_a^b f(x)dx = \int_{-1}^1 \frac{dx}{1 + x^2} = \left[\text{arc tg}\right]_{-1}^1 = \frac{\pi}{2}. \qquad (3$$

Let us change the variable, setting

$$x = \frac{1}{u} = \psi(u).$$

Then

$$a = -1, \quad b = 1; \quad \alpha = -1, \quad \beta = 1.$$

Also

$$\int_{-1}^1 f[\psi(u)]\psi'(u)du = -\int_{-1}^1 \frac{du}{1 + u^2} = -\frac{\pi}{2}. \qquad (4$$

The two integrals 3), 4) are not equal. The reason for this is that the function

$$u = \phi(x) = \frac{1}{x}$$

of 540 does not have a continuous derivative in $\mathfrak{A} = (-1, 1)$. Indeed, it is not even defined throughout $\mathfrak{A}$.

542. *Let $x = \psi(u)$ have a continuous derivative in $\mathfrak{B} = (\alpha, \beta)$, $\alpha \lessgtr \beta$. Let $\mathfrak{A}$ be the image of $\mathfrak{B}$. Let $f(x)$ be limited and integrable in $\mathfrak{A}$, and let $f[\psi(u)]\psi'(u)$ be integrable in $\mathfrak{B}$. Then*

$$\int_a^b f(x)dx = \int_\alpha^\beta f[\psi(u)]\psi'(u)du. \qquad a = \psi(\alpha), \ b = \psi(\beta). \qquad (1$$

1. Let us note first the difference between this theorem and that of 540. In 540 $\psi(u)$ is univariant, and $\mathfrak{A}$, $\mathfrak{B}$ are in uniform correspondence.

In the present theorem, ψ may have any number of oscillations in $\mathfrak{B}$. Furthermore, the intervals $\mathfrak{A}$ and (a, b) may not be the same.

Example. Let $x = \psi(u) = \sin u$, $\mathfrak{B} = (0, \frac{13}{6}\pi)$. Then the image of $\mathfrak{B}$ is the interval $\mathfrak{A} = (-1, 1)$. On the other hand, $a = \sin 0 = 0$, $b = \sin \frac{13}{6}\pi = \frac{1}{2}$. Thus $(a, b) = (0, \frac{1}{2})$ is different from $\mathfrak{A}$.

Let $f(x) = x$. Then

$$\int_a^b f dx = \int_0^{\frac{1}{2}} x\, dx = \tfrac{1}{8}\,; \quad \int_a^\beta f(\psi u)\psi' u\, du = \int_0^{\frac{13}{6}\pi} \sin u \cos u\, du = \tfrac{1}{8}.$$

Thus the two integrals are equal, as the theorem requires.

2. To prove the formula 1), consider

$$F(u) = \int_{\psi(a)}^{\psi(u)} f(x)dx - \int_a^u g(u)du, \qquad g(u) = f[\psi(u)]\psi'(u).$$

We shall show that $F(u)$ is a constant in $\mathfrak{B}$. As it is 0 at α, $F = 0$ throughout $\mathfrak{B}$.

To this end we show

$$F'(u) = 0, \qquad \text{in } \mathfrak{B}.$$

Then, by 400, 2, $F(u)$ is a constant in $\mathfrak{B}$, and therefore 0.

At any point u of $\mathfrak{B}$, we have

$$\Delta F = \int_{\psi(u)}^{\psi(u)+\Delta\psi} f(x)dx - \int_u^{u+\Delta u} g(u)du. \tag{2}$$

There are two cases:

1°. $\psi'(u) \neq 0$. Then, by 403, $\psi(u)$ is univariant in $V(u)$. We can thus apply 540. Hence

$$\Delta F = 0, \qquad \text{in } V(u),$$

and therefore

$$F' = 0, \qquad \text{at } u.$$

2°. $\psi'(u) = 0$. Let us apply the theorem of the mean 532 to each integral in 2). Then

$$\Delta F = \Phi\Delta\psi - \Psi\Delta u,$$

where

$$\Phi = \text{Mean } f(x), \quad \Psi = \text{Mean } g(u)$$

in $\Delta\psi$, Δu respectively. Thus

$$\frac{\Delta F}{\Delta u} = \Phi\frac{\Delta\psi}{\Delta u} - \Psi.$$

Now
$$\lim \frac{\Delta\psi}{\Delta u} = \psi'(u) = 0.$$

Also
$$\lim \Psi = 0,$$

since $f[\psi(u)]$ is limited in $\mathfrak{A}$, and

$$\lim_{v=u} \psi'(v) = \psi'(u) = 0,$$

ψ' being continuous. Hence $F'(u) = 0$, also in this case.

543. *Let $f(x)$ be limited in $\mathfrak{A} = (a, b)$, $a \gtrless b$. Let $u = \phi(x)$ have a continuous derivative $\phi'(x) \neq 0$ in $\mathfrak{A}$. Let $\mathfrak{B} = (\alpha, \beta)$ be the image of $\mathfrak{A}$; $\alpha = \phi(a)$, $\beta = \phi(b)$. Let $x = \psi(u)$ be the inverse function of ϕ. Then*

$$\overline{\int_a^b} f dx = \overline{\int_\alpha^\beta} f(\psi(u))\psi'(u) du, \tag{1}$$

$$\underline{\int_a^b} f dx = \underline{\int_\alpha^\beta} f(\psi(u))\psi'(u) du. \tag{2}$$

Let us prove 1); the demonstration of 2) is similar. Since ϕ is univariant, the intervals $\mathfrak{A}$ and $\mathfrak{B}$ stand in uniform correspondence by 358. To fix the ideas let ϕ be an increasing function.

Then by 381, $\psi'(u) > 0$, and continuous. Let $E = (u_1, u_2, \cdots)$ be a division of $\mathfrak{B}$ of norm δ into subintervals Δu_κ, to which corresponds a division $D = (x_1, x_2, \cdots)$ of $\mathfrak{A}$ of norm d into intervals Δx_κ. Let

$$L_\kappa = \operatorname{Max} f(x), \qquad \text{in } \Delta x_\kappa.$$

$$= \operatorname{Max} f(\psi(u)), \qquad \text{in } \Delta u_\kappa.$$

$$M_\kappa = \operatorname{Max} f(\psi(u))\psi'(u), \qquad \text{in } \Delta u_\kappa.$$

$$\lambda_\kappa = \operatorname{Min} \psi'(u), \qquad \mu_\kappa = \operatorname{Max} \psi'(u), \qquad \text{in } \Delta u_\kappa.$$

$$F = \operatorname{Max} |f|, \qquad \text{in } \mathfrak{A}.$$

We have to show that

$$\bar{S}_D = \Sigma L_\kappa \Delta x_\kappa, \qquad \bar{S}_E = \Sigma M_\kappa \Delta u_\kappa$$

have the same limits.

Since $\phi'(x)$ and $\psi'(u)$ are continuous, they are uniformly continuous. Hence d and δ converge to 0 simultaneously. For this reason for any $\delta \leq$ some δ_0,

$$\mu_\kappa - \lambda_\kappa < \frac{\epsilon}{F|\beta - \alpha|}, \qquad \text{uniformly in } \mathfrak{B}.$$

By the Law of the Mean,

$$\Delta x_\kappa = \psi'(v_\kappa)\Delta u_\kappa, \qquad v_\kappa \text{ in } \Delta u_\kappa.$$

Hence

$$\Theta = \bar{S}_E - \bar{S}_D = \Sigma\{M_\kappa - L_\kappa\psi'(v_\kappa)\}\Delta u_\kappa.$$

But, obviously,

$$\text{Max} f \text{ Min } \psi' \leq \text{Max} f\psi' \leq \text{Max} f \text{ Max } \psi';$$

if Max $f > 0$, while the signs are reversed if it is < 0.

Thus in either case M_κ lies between $L_\kappa\lambda_\kappa$ and $L_\kappa\mu_\kappa$. Also, $L_\kappa\psi'(v_\kappa)$ lies between these same bounds. Hence

$$|M_\kappa - L_\kappa\psi'(v_\kappa)| \leq L_\kappa(\mu_\kappa - \lambda_\kappa) < \frac{\epsilon}{|\beta - \alpha|}.$$

Hence

$$|\Theta| < \frac{\epsilon|\Sigma\Delta u_\kappa|}{|\beta - \alpha|} = \epsilon. \qquad \delta < \delta_0.$$

544. *Let $x = \psi(u)$ have a continuous derivative in $\mathfrak{B} = (\alpha, \beta)$, $\alpha \gtrless \beta$. Let ψ' vanish over a discrete aggregate Δ, but otherwise let it have one sign. Let $\mathfrak{A} = (a, b)$ be the image of $\mathfrak{B}$, $a = \psi(\alpha)$, $b = \psi(\beta)$. If one of the integrals*

$$X = \int_a^b f dx = \int_{\mathfrak{A}},$$

$$U = \int_\alpha^\beta f[\psi(u)]\psi'(u)du = \int_{\mathfrak{B}},$$

exists, the other does, and both are then equal.

To fix the ideas let U exist. By 403 the correspondence between $\mathfrak{A}$, $\mathfrak{B}$ is uniform.

Let us effect a division, of norm δ, of $\mathfrak{B}$. Let the norm of the corresponding division of $\mathfrak{A}$ be η. Let $\mathfrak{B}_1$ be those intervals containing no points of Δ, while $\mathfrak{B}_2 = \mathfrak{B} - \mathfrak{B}_1$ is the complement of $\mathfrak{B}_1$.

Let $\mathfrak{A}_1$, $\mathfrak{A}_2$ correspond to $\mathfrak{B}_1$, $\mathfrak{B}_2$ respectively.

Now

$$\int_{\mathfrak{B}} = \int_{\mathfrak{B}_1} + \int_{\mathfrak{B}_2}. \tag{1}$$

But, by 543,

$$\int_{\mathfrak{B}_1} = \int_{\mathfrak{A}_1},$$

while, since Δ is discrete,

$$\lim_{\delta=0} \int_{\mathfrak{B}_2} = 0.$$

Hence 1) gives

$$\int_{\mathfrak{B}} = \lim_{\delta=0} \int_{\mathfrak{A}_1} = \lim_{\eta=0} \int_{\mathfrak{A}_1}$$

$$= \int_{\mathfrak{A}}, \quad \text{by 522, 3.}$$

A similar reasoning holds, if we assume that X exists.

Second Theorem of the Mean

545. *Let $f(x)$ be limited and integrable in $\mathfrak{A} = (a, b)$. Let $g(x)$ be limited and monotone in $\mathfrak{A}$. Then*

$$\int_a^b fg dx = g(a+0) \int_a^{\xi} f dx + g(b-0) \int_{\xi}^b f dx, \quad a \leq \xi \leq b. \tag{1}$$

Since g is limited and monotone, it is integrable in $\mathfrak{A}$ by 502. Hence fg is integrable, by 505.

By 277, 8,

$$g(a+0), \; g(b-0) \text{ exist.}$$

If $g(a+0) = g(b-0)$, the relation 1) is obviously true. We therefore assume that these limits are different.

To fix the ideas, let $g(x)$ be monotone increasing.

We begin by effecting a division of $\mathfrak{A}$,

$$D(a_1, a_2 \cdots a_{n-1}),$$

of norm δ. We also set

$$a = a_0, \qquad b = a_n.$$

Let

$$M_\kappa = \text{Max} f, \qquad m_\kappa = \text{Min} f,$$

in the interval

$$\delta_\kappa = (a_{\kappa-1} a_\kappa).$$

Let ξ_κ be any point in δ_κ.

From

$$M_\kappa \geq f(\xi_\kappa) \geq m_\kappa,$$

we have

$$M_\kappa \delta_\kappa \geq f(\xi_\kappa)\delta_\kappa \geq m_\kappa \delta_\kappa, \tag{2}$$

and also

$$M_\kappa \delta_\kappa \geq \int_{\delta_\kappa} f(\xi_\kappa) dx \geq m_\kappa \delta_\kappa. \tag{3}$$

From 2), 3) we have

$$|f(\xi_\kappa)\delta_\kappa - \int_{\delta_\kappa}| \leq (M_\kappa - m_\kappa)\delta_\kappa.$$

Hence

$$f(\xi_\kappa)\delta_\kappa = \int_{\delta_\kappa} f dx + \sigma_\kappa, \tag{4}$$

$$|\sigma_\kappa| \leq (M_\kappa - m_\kappa)\delta_\kappa.$$

Multiply 4) by $g(\xi_\kappa)$; and letting $\kappa = 1, 2, \cdots n$, let us sum the n resulting equations. We get

$$\Sigma f(\xi_\kappa) g(\xi_\kappa)\delta_\kappa = \Sigma g(\xi_\kappa) \int_{\delta_\kappa} f dx + \Sigma \sigma_\kappa g(\xi_\kappa). \tag{5}$$

Now

$$\int_{\delta_\kappa} = \int_{a_{\kappa-1}}^b - \int_{a_\kappa}^b,$$

or more briefly,

$$= \int_{\kappa-1} - \int_\kappa.$$

Hence letting $\kappa = 1, 2, \cdots n$, we get

$$g(\xi_1)\int_{\delta_1} = g(\xi_1)\int_0 - g(\xi_1)\int_1$$

$$g(\xi_2)\int_{\delta_2} = g(\xi_2)\int_1 - g(\xi_2)\int_2$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$g(\xi_{n-1})\int_{\delta_{n-1}} = g(\xi_{n-1})\int_{n-2} - g(\xi_{n-1})\int_{n-1}$$

$$g(\xi_n)\int_{\delta_n} = g(\xi_n)\int_{n-1}.$$

Adding, we have,

$$\sum g(\xi_\kappa)\int_{\delta_\kappa} f dx = g(\xi_1)\int_a^b f dx + \sum_{\kappa=2}^n \left\{ g(\xi_\kappa) - g(\xi_{\kappa-1}) \right\} \int_{\kappa-1} f dx. \tag{6}$$

Since g is monotone increasing,

$$G_\kappa = g(\xi_\kappa) - g(\xi_{\kappa-1}) \geqq 0.$$

Let $\mathfrak{M}$ be the maximum of the integral $\int_x^b f dx$, and $\mathfrak{m}$ its minimum as x ranges over $\mathfrak{A}$. Then

$$G_\kappa \mathfrak{m} \leq \left\{ g(\xi_\kappa) - g(\xi_{\kappa-1}) \right\} \int_{\kappa-1} \leq G_\kappa \mathfrak{M},$$

and adding,

$$\mathfrak{m} \sum G_\kappa \leq \sum \left\{ g(\xi_\kappa) - g(\xi_{\kappa-1}) \right\} \int_{\kappa-1} \leq \mathfrak{M} \sum G_\kappa. \qquad (7$$

But

$$\Sigma G_\kappa = g(\xi_n) - g(\xi_1).$$

Thus 7) gives

$$\sum \left\{ g(\xi_\kappa) - g(\xi_{\kappa-1}) \right\} \int_{\kappa-1} f dx = \Theta \left\{ g(\xi_n) - g(\xi_1) \right\}, \qquad (8$$

where

$$\mathfrak{m} \leq \Theta \leq \mathfrak{M}. \qquad (9$$

Thus 5), 6), 8) give

$$\sum f(\xi_\kappa) g(\xi_\kappa) \delta_\kappa = g(\xi_1) \int_a^b f dx + \Theta \left\{ g(\xi_n) - g(\xi_1) \right\} + \sum \sigma_\kappa g(\xi_\kappa). \qquad (10$$

In this equation let $\delta \doteq 0$. The limit of the left side is

$$\int_a^b fg dx.$$

Also

$$\lim g(\xi_1) = g(a+0) ; \quad \lim g(\xi_n) = g(b-0).$$

Let

$$|g(x)| < \Gamma ;$$

then

$$|\Sigma \sigma_\kappa g(\xi_\kappa)| < \Gamma \Sigma \sigma_\kappa.$$

But

$$\Sigma \sigma_\kappa \leqq \underset{D}{\Omega} f, \quad \text{by } 494.$$

As f is integrable,

$$\lim \underset{D}{\Omega} f = 0 ;$$

hence the last term of 10) has the limit 0.

Thus all the terms of 10), besides that in Θ, have finite limits. Hence the limit of Θ exists. Call it θ.

Hence

$$\mathfrak{m} \leq \theta \leq \mathfrak{M}.$$

But F being a continuous function of x, it takes on the value θ for some point ξ in $\mathfrak{A}$ by 357.

Hence

$$\theta = \int_\xi^b f dx.$$

Passing now to the limit in 10), we have

$$\int_a^b fg dx = g(a+0)\int_a^b f dx + \left\{ g(b-0) - g(a+0) \right\} \int_\xi^b f dx. \quad (11$$

But since

$$\int_a^b = \int_a^\xi + \int_\xi^b,$$

we get 1) at once from 11).

546. If $g(x)$ is not monotone, the formula 1) of 545 may not be true, as the following example shows.

Let

$$f(x) = x^2, \qquad g(x) = \cos x.$$

Then

$$\int_{-\pi/2}^{\pi/2} x^2 \cos x \, dx > 0, \qquad \text{by 527, 1,}$$

since the integrand is never negative and is in general positive. On the other hand,

$$g(a+0) = \cos - \pi/2 = 0 ; \qquad g(b-0) = \cos \pi/2 = 0.$$

Hence the right side of 1), 545, is zero. The formula 1) is thus untrue in this case.

INDEFINITE INTEGRALS

Primitive Functions

547. 1. The theorem of 538 is of great importance in evaluating integrals. For, to find the value of

$$J = \int_a^x f dx, \quad (1$$

$f(x)$ being limited and integrable in $\mathfrak{A} = (\alpha, x)$, we have only to seek a function $F(x)$ which is one-valued in $\mathfrak{A}$ and has $f(x)$ as derivative. Then, as we saw,

$$J = F(x) - F(\alpha). \quad (2$$

Let $G(x)$ be any other function which is one-valued in $\mathfrak{A}$ and has $f(x)$ as derivative. Then

$$J = G(x) - G(\alpha). \qquad (3$$

Comparing 2), 3), we have

$$G(x) = F(x) + C,$$

where C is a constant.

The functions $F(x)$, $G(x)$ are called *primitive functions of* $f(x)$. They are denoted by

$$\int f(x)dx,$$

no limits of integration appearing in the symbol. Primitive functions are also called *indefinite integrals*. In contradistinction, integrals of the type 1) are called *definite integrals*.

2. Every formula of *differentiation*, as,

$$D_x F(x) = f(x),$$

$f(x)$ being limited and integrable in an interval $\mathfrak{A}$, gives rise to a formula of *integration*,

$$\int f(x)dx = F(x).$$

For reference, we add a short table of indefinite integrals.*

We observe, once for all, that in all formulæ involving indefinite integrals, we shall suppose that the integrands are one-valued, limited, and integrable in a certain interval $\mathfrak{A}$, while the functions outside the integral sign are one-valued in $\mathfrak{A}$.

548.

$$\int a dx = ax.$$

$$\int x^\mu dx = \frac{x^{\mu+1}}{\mu+1}, \qquad \mu \neq -1.$$

$$\int \frac{dx}{x} = \log |x|.$$

$$\int e^x dx = e^x.$$

---

* An excellent table of integrals is "A Short Table of Integrals" by B. O. Peirce. Ginn & Co. Boston.

$$\int a^x dx = \frac{a^x}{\log a}, \qquad a > 0.$$

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \text{ arc tg } \frac{x}{a}, \qquad a \neq 0.$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \text{arc sin } \frac{x}{a}, \qquad a \neq 0.$$

$$\int \sin x \, dx = - \cos x.$$

$$\int \cos x \, dx = \sin x.$$

$$\int \tan x \, dx = - \log |\cos x|.$$

$$\int \cot x \, dx = \log |\sin x|.$$

$$\int \tan x \sec x \, dx = \sec x.$$

$$\int \sec^2 x \, dx = \tan x.$$

549. Not every limited integrable function in $\mathfrak{A} = (a, b)$ has a primitive, as we now show.

Let $f(x)$ be continuous in $\mathfrak{A}$; let

$$F(x) = \int_a^x f dx, \qquad a \leq x \leq b.$$

Then, by 537,

$$\frac{dF}{dx} = f(x), \qquad \text{in } \mathfrak{A}.$$

Let us define a limited function $g(x)$ in $\mathfrak{A}$ as follows: it shall $= f(x)$ except at points of a discrete aggregate in Δ, at which points $f(x) \neq g(x)$. Then, by 530,

$$\int_a^x g dx = F(x). \tag{1}$$

Suppose now $g(x)$ had a primitive $G(x)$ in $\mathfrak{A}$. Then, by 538,

$$\int_a^x g dx = G(x) - G(a). \tag{2}$$

Comparing 1), 2), we get

$$G(x) = F(x) + C, \qquad C \text{ a constant.}$$

Hence

$$\frac{dG}{dx} = \frac{dF}{dx}, \qquad \text{in } \mathfrak{A};$$

or

$$f(x) = g(x), \qquad \text{in } \mathfrak{A},$$

which is a contradiction.

Methods of Integration

550. In order to find the primitive of a function $f(x)$ we may proceed as follows: We first consult a table of indefinite integrals. If the integral we are seeking is not there, we try to transform it into one or more integrals which are in the table.

The principal transformations employed are:

1°. Decomposition of the integrand into a sum.
2°. Integration by parts.
3°. Change of variable.

We treat these now separately.

551. *Decomposition of the integrand into a sum.* This method, as its name implies, consists in breaking $f(x)$ up into a sum of simpler functions. Thus, if

$$f(x) = f_1(x) + \cdots + f_s(x),$$

then

$$\int f dx = \int f_1 dx + \cdots + \int f_s dx.$$

Ex. **1.**

$$J = \int \cos^2 x\, dx.$$

As

$$\cos^2 x = \frac{1 + \cos 2x}{2},$$

$$J = \tfrac{1}{2}\int dx + \tfrac{1}{2}\int \cos 2x\, dx$$

$$= \tfrac{1}{2}x + \tfrac{1}{4}\sin 2x.$$

Ex. 2.

$$J=\int\frac{a+bx}{\alpha+\beta x}dx.$$

Since

$$\frac{a+bx}{\alpha+\beta x}=\frac{b}{\beta}+\frac{a\beta-\alpha b}{\beta(\alpha+\beta x)},\qquad \text{by 91, 2)},$$

$$J=\frac{b}{\beta}\int dx+\frac{a\beta-\alpha b}{\beta}\int\frac{dx}{\alpha+\beta x}.$$

Now

$$\frac{dx}{\alpha+\beta x}=\frac{1}{\beta}\,\frac{d\cdot(\alpha+\beta x)}{\alpha+\beta x}=\frac{1}{\beta}\,d\cdot\log(\alpha+\beta x).$$

Hence

$$\int\frac{dx}{\alpha+\beta x}=\frac{1}{\beta}\log(\alpha+\beta x).$$

Thus

$$J=\frac{b}{\beta}x+\frac{a\beta-\alpha b}{\beta^2}\log(\alpha+\beta x).$$

Integration by Parts

552. In the interval $\mathfrak{A}$, let $u(x)$, $v(x)$ be one-valued functions having limited integrable derivatives. Then

$$D_x uv=uv'+vu'.$$

Hence

$$\int uv'\,dx=uv-\int vu'\,dx. \tag{1}$$

The application of 1) to evalue

$$J=\int f dx,$$

is as follows. We write

$$f=uv'.$$

Then 1) shows that

$$J=uv-\int vu'\,dx.$$

The determination of J is thus made to depend upon

$$\int vu'\,dx.$$

553. Ex. 1.

$$J=\int x\log x\,dx.$$

Set

$$u=\log x,\qquad v'=x.$$

Then
$$u' = \frac{1}{x}, \qquad v = \tfrac{1}{2}x^2.$$

Hence
$$J = \tfrac{1}{2}x^2 \log x - \tfrac{1}{2}\int x\,dx$$
$$= \frac{x^2}{2}(\log x - \tfrac{1}{2}).$$

554. Ex. 2.
$$J = \int e^{ax} \sin bx\,dx, \qquad a,\ b \neq 0.$$

Set
$$u = \sin bx. \qquad v' = e^{ax}.$$

Then
$$u' = b\cos bx, \qquad v = \frac{1}{a}e^{ax}.$$

Hence
$$J = \frac{1}{a}e^{ax}\sin bx - \frac{b}{a}\int e^{ax}\cos bx\,dx. \tag{1}$$

To find
$$K = \int e^{ax}\cos bx\,dx,$$

set
$$u = \cos bx, \qquad v' = e^{ax}.$$

Then
$$u' = -b\sin bx, \qquad v = \frac{1}{a}e^{ax}.$$

Hence
$$K = \frac{1}{a}e^{ax}\cos bx + \frac{b}{a}\int e^{ax}\sin bx\,dx$$
$$= \frac{1}{a}e^{ax}\cos bx + \frac{b}{a}J.$$

This placed in 1) gives
$$J = \frac{e^{ax}\{a\sin bx - b\cos bx\}}{a^2+b^2}. \tag{2}$$

The same method gives
$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}\{a\cos bx + b\sin bx\}}{a^2+b^2}. \tag{3}$$

Change of Variable

555. *Let $f(x)$ be continuous in the interval $\mathfrak{A}$. Let $u = \phi(x)$ have a continuous derivative $\phi'(x) \neq 0$ in $\mathfrak{A}$. Let $\mathfrak{B}$ be the image of $\mathfrak{A}$, and $x = \psi(u)$ be the inverse function of ϕ. Then if*
$$\int f[\psi(u)]\psi'(u)du = G(u), \qquad \textit{valid in } \mathfrak{B};$$

we have
$$\int f(x)dx = G[\phi(x)], \qquad \textit{valid in } \mathfrak{A}.$$

For,
$$\frac{dG[\phi(x)]}{dx} = \frac{dG(u)}{du}\cdot\frac{du}{dx} \tag{1}$$

But by hypothesis,

$$\frac{dG}{du}=f[\psi(u)]\psi'(u)=f[\psi(u)]\frac{dx}{du}.$$

As

$$\frac{du}{dx}\cdot\frac{dx}{du}=1,$$

1) gives

$$\frac{dG[\phi(x)]}{dx}=f[\psi(u)]=f(x).$$

556. Ex. 1.

$$J=\int\frac{dx}{x\log x}.$$

Set

$$u=\log x.$$

Then

$$J=\int\frac{du}{u}=\log u=\log\log x.$$

557. Ex. 2.

$$J=\int\frac{dx}{x\sqrt{2ax-a^2}}.$$

Set

$$u=\sqrt{2ax-a^2}.$$

Then

$$J=2\int\frac{du}{a^2+u^2}=\frac{2}{a}\operatorname{arc\,tg}\frac{u}{a}$$

$$=\frac{2}{a}\operatorname{arc\,tg}\frac{\sqrt{2ax-a^2}}{a}.$$

558. Ex. 3.

$$J=\int\frac{dx}{\sqrt{x^2\pm a^2}}.$$

Set

$$u=x+\sqrt{x^2\pm a^2}.$$

Then

$$J=\int\frac{du}{u}=\log u$$

$$=\log(x+\sqrt{x^2\pm a^2}).$$

559. Ex. 4.

$$J=\int\frac{dx}{\sqrt{(x-a)(x-b)}}.$$

Set

$$u=\sqrt{x-a}.$$

Then

$$J=2\int\frac{du}{\sqrt{u^2+a-b}},$$

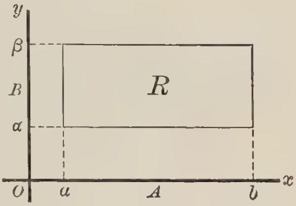
an integral evaluated in Ex. 3.

INTEGRALS DEPENDING ON A PARAMETER

560. Let the rectangle bounded by the lines $x = a$, $x = b$, $y = \alpha$, $y = \beta$ be denoted by

$$R = (a, b, \alpha, \beta).$$

Let $\mathfrak{A} = (a, b)$, $\mathfrak{B} = (\alpha, \beta)$.



Let $f(x, y)$ be defined over R. If we give to y an *arbitrary* but *fixed* value in $\mathfrak{B}$, $f(x, y)$ is a function of x defined over $\mathfrak{A}$. But since its value also depends on the particular value assigned to y, we say f is a function of x which depends upon the *parameter* y. If for each value of y in $\mathfrak{B}$, $f(x, y)$ is a limited integrable function of x in $\mathfrak{A}$, we shall say it is *regular* in R. When $f(x, y)$ is regular in R,

$$J(y) = \int_a^b f(xy)dx \tag{1}$$

defines a one-valued function of y, over the interval $\mathfrak{B}$.

In performing the integration indicated in 1), we consider y as constant and integrate with respect to x.

In the present section we propose to study the function $J(y)$ with respect to *continuity*, *differentiation*, and *integration*.

Ex. 1.

$$J(y) = \int_0^\pi \frac{\sin xy}{e^x} dx.$$

$\mathfrak{A} = (0, \pi)$, $\mathfrak{B}$ any interval.

Ex. 2.

$$J(y) = \int_0^\pi \log(1 - 2y\cos x + y^2)dx.$$

$\mathfrak{A} = (0, \pi)$, $\mathfrak{B}$ any interval not including $y = \pm 1$.
For, $\log u$ is continuous when $u > 0$. Here

$$u = 1 - 2y\cos x + y^2 = (y - \cos x)^2 + \sin^2 x \geqq 0,$$

and hence $u = 0$ only for the points whose coördinates are

$$x = m\pi,\ y = (-1)^m. \tag{2}$$

Continuity

561. Let η be an arbitrary but fixed value of y in $\mathfrak{B} = (\alpha, \beta)$.

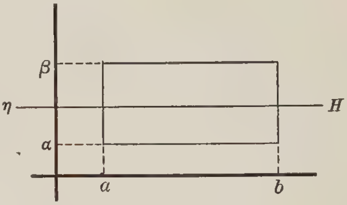
Let us denote the line $y = \eta$ by H.

Let $\phi(x)$ be defined over $\mathfrak{A} = (a, b)$.

If for each $\epsilon > 0$, there exists a $\delta > 0$, such that

$$|f(x, \eta + h) - \phi(x)| < \epsilon, \qquad (1$$

for each $0 < |h| < \delta$, and every x in $\mathfrak{A}$, we say: *$f(x, y)$ converges uniformly to $\phi(x)$ along the line H, or with respect to the line H.*



We denote this by

$$\lim_{y=\eta} f(x, y) = \phi(x), \qquad \textit{uniformly};$$

or

$$f(x, y) \doteq \phi(x), \qquad \textit{uniformly along } H.$$

If in the relation 1), only positive values of h are considered, we say $f(x, y)$ converges *on the right* uniformly, etc.

If only negative values of h are considered, $f(xy)$ converges *on the left* uniformly, etc.

If $f(xy)$ converges uniformly to $f(x, \eta)$ with respect to the line H, we shall say $f(xy)$ is a *uniformly continuous function of y with respect to the line H*, or *along the line H.*

If $f(xy)$ is a uniformly continuous function of y with respect to each line $y = \eta$ in $\mathfrak{B} = (\alpha\beta)$, we shall say *$f(xy)$ is a uniformly continuous function of y in $\mathfrak{B}$.*

562. 1. *Let $f(x, y)$ be regular in any $R = (ab\gamma\beta)$, $\alpha < \gamma < \beta$. Let*

$$R \lim_{y=\alpha} f(x, y) = \phi(x)$$

uniformly along the line $y = \alpha$.

Let $\phi(x)$ be limited and integrable in $\mathfrak{A}$.

Then

$$R \lim_{y=\alpha} \int_a^b f(xy)dx = \int_a^b R \lim_{y=\alpha} f(xy)dx = \int_a^b \phi(x)dx. \qquad (1$$

For, let

$$\Delta = \int_a^b f(x, \alpha + h)dx - \int_a^b \phi(x)dx = \int_a^b \{f(x, \alpha + h) - \phi(x)\}dx. \qquad (2$$

We have to show that

$$\epsilon > 0, \quad \delta > 0, \quad |\Delta| \leq \epsilon, \quad 0 < h < \delta. \tag{3}$$

But by hypothesis, for each $\epsilon > 0$, there exists a $\delta > 0$, such that

$$|f(x, \alpha + h) - \phi(x)| < \frac{\epsilon}{b - a} \tag{4}$$

for each $0 < h < \delta$, and any x in $\mathfrak{A}$.

Hence 3) follows from 2), 4), and 524, 2).

2. That the relation 1) may not hold when $f(x, y)$ does not converge uniformly to $\phi(x)$, is shown by the following example:

Let

$$f(x, y) = \frac{y}{x^2 + y^2}, \quad \text{for } x \neq 0;$$

$$= 0, \quad \text{for } x = 0.$$

Here

$$R \lim_{y=0} f(x, y) = \phi(x) = 0.$$

Hence

$$\int_0^1 \phi(x) dx = 0. \tag{5}$$

On the other hand,

$$J = \int_0^1 f dx = \int_0^1 \frac{y}{x^2 + y^2} dx = \text{arc tg} \frac{1}{y}, \quad y > 0.$$

Hence

$$R \lim_{y=0} J = \frac{\pi}{2}. \tag{6}$$

As 5), 6) have different values, the relation 1) does not hold here. Obviously $f(x, y)$ does not converge uniformly to 0 in any interval containing the origin.

563. 1. As corollaries of 562 we have:

Let $f(x, y)$ be regular in $R = (ab\alpha\beta)$. Let it be uniformly continuous in y, along the line $y = \eta$. Then

$$J(y) = \int_a^b f(x, y) dx$$

is a continuous function of y at η. $\alpha \leq \eta \leq \beta$.

2. *Let $f(x, y)$ be regular in $R = (ab\alpha\beta)$. Let it be a uniformly continuous function of y in $\mathfrak{B}$. Then $J(y)$ is continuous in $\mathfrak{B}$.*

3. *If $f(x, y)$ be continuous in $R(ab\alpha\beta)$, $J(y)$ is continuous in $\mathfrak{B} = (\alpha, \beta)$.*

This follows at once from 352.

Example. In 538 we proved the relation

$$\phi(\lambda) = \int_0^{\pi/2} \frac{dx}{1 + \lambda^2 \tan^2 x} = \frac{\pi}{2} \cdot \frac{1}{1 + |\lambda|} \tag{1}$$

for all values of λ. It required, however, a separate integration to establish it for $\lambda = \pm 1$. By the aid of 3, we may prove the correctness of 1) for these values without any calculation. Consider, to fix the ideas, $\lambda = 1$. Since the integrand of 1) is obviously a continuous function of x, λ in the band $R = (0, \pi/2, 1 - \delta, 1 + \delta)$, the integral is a continuous function of λ at 1. Hence 1) holds for $\lambda = 1$.

564. The results of 562, 563 may be generalized as follows:

Let Δ be a discrete point aggregate in $\mathfrak{A}$. We can divide $\mathfrak{A}$ into two systems of intervals, $\mathfrak{C}$ and $\mathfrak{D}$, such that $\mathfrak{C}$ contains no point of Δ, and the total length d of the intervals $\mathfrak{D}$ is as small as we please.

We shall say $f(x, y)$ converges uniformly to $\phi(x)$ along the line $y = \eta$, *except at the points* Δ, when, for each $\epsilon > 0$ and any $\mathfrak{C}$, there exists a $\delta > 0$, such that

$$|f(x, \eta + h) - \phi(x)| < \epsilon$$

for each $0 < |h| < \delta$ and every x in $\mathfrak{C}$.

The terms, $f(x, y)$ converges *on the right*, or *on the left uniformly, except for the points* Δ, need no special explanation.

Also the meaning of the term $f(x, y)$ is *uniformly continuous* along the line $y = \eta$, *except for the points* Δ, is obvious.

565. *Let $f(x, y)$ be regular in the rectangle $R(a, b, \alpha, \beta)$. Let f converge uniformly to $\phi(x)$ along the line $y = \eta$, except for the points of a discrete aggregate Δ. Let $\phi(x)$ be limited and integrable in $\mathfrak{A} = (a, b)$. Then*

$$\lim_{y=\eta} \int_a^b f(x, y)dx = \int_a^b \lim_{y=\eta} f(x, y)dx = \int_a^b \phi(x)dx, \qquad \alpha \leq \eta \leq \beta.$$

Let

$$D = \int_a^b f(x, \eta + h)dx - \int_a^b \phi(x)dx.$$

We must show that

$$\epsilon > 0, \qquad \delta > 0, \qquad |D| < \epsilon, \qquad 0 < |h| < \delta. \tag*{(1}$$

Since f is limited in R, and ϕ in $\mathfrak{A}$,

$$|\phi(x)|, \qquad |f(x, y)| < M, \qquad \text{in } R.$$

Choosing $\epsilon > 0$ small at pleasure, and then fixing it, we choose the system $\mathfrak{D}$ such that its length

$$d < \frac{\epsilon}{4M}.$$

Then

$$D = \int_{\mathfrak{C}} \{f(x, \eta + h) - \phi(x)\} dx + \int_{\mathfrak{D}} \{f(x, \eta + h) - \phi(x)\} dx.$$

Hence

$$|D| \le \left|\int_{\mathfrak{C}}\right| + \left|\int_{\mathfrak{D}}\right|. \tag*{(2}$$

But

$$\left|\int_{\mathfrak{D}}\right| < \frac{\epsilon}{2}.$$

On the other hand, by hypothesis,

$$|f(x, \eta + h) - \phi(x)| < \frac{\epsilon}{2(b-a)}$$

for each $0 < |h| < \delta$, and every x in $\mathfrak{C}$. Hence

$$\left|\int_{\mathfrak{C}}\right| \le \frac{\epsilon}{2}.$$

Hence 2) gives

$$|D| < \frac{\epsilon}{2} + \frac{\epsilon}{2},$$

which proves 1).

566. As corollary we have:

Let $f(x, y)$ be regular in the rectangle $R = (a, b, \alpha, \beta)$. Let it be uniformly continuous in y along the line $y = \eta$, except for the points of a discrete aggregate Δ. Then

$$J(y) = \int_a^b f(x, y) dx$$

is continuous at η, $\qquad \alpha \le \eta \le \beta.$

Differentiation

567. 1. 1°. *Let $f(x, y), f'_y(x, y)$ be regular in $R = (ab\alpha\beta)$.*

2°. *Let f'_y be uniformly continuous in y along the line $y = \eta$, except for the points of a discrete aggregate.*

Let

$$J(y) = \int_a^b f(x, y)dx.$$

Then

$$J'(\eta) = \int_a^b f'_y(x, \eta)dx, \qquad \alpha \leq \eta \leq \beta. \tag{1}$$

For,

$$\frac{\Delta J}{\Delta \eta} = \frac{J(\eta + h) - J(\eta)}{h} = \int_a^b \frac{f(x, \eta + h) - f(x, \eta)}{h} dx. \tag{2}$$

But by the Law of the Mean,

$$\frac{f(x, \eta + h) - f(x, \eta)}{h} = f'_y(x, \zeta),$$

where ζ lies between η and $\eta + h$ and depends on x and h. But by 2°,

$$f'_y(x, \zeta) = f'_y(x, \eta) + \sigma;$$

where

$$\lim_{h=0} \sigma = 0, \qquad \textit{uniformly except for } \Delta.$$

Thus

$$\frac{\Delta J}{\Delta \eta} = \int_a^b f'_y(x, \eta)dx + \int_a^b \sigma dx, \tag{3}$$

which gives 1), on passing to the limit, $h = 0$.

2. As corollary we have:

Let $f(x, y), f'_y(x, y)$ be continuous in the rectangle $(ab\alpha\beta)$. Then

$$\frac{d}{dy}\int_a^b f(x, y)dx = \int_a^b f'_y(x, y)dx. \qquad \alpha \leq y \leq \beta. \tag{3}$$

3. *Criticism.* Many text-books give the following incorrect demonstration of 1. From 2) we have, changing slightly the notation,

$$\frac{dJ}{dy} = \lim \frac{\Delta J}{\Delta y} = \lim \int_a^b \frac{\Delta f}{\Delta y}. \tag{4}$$

It is now assumed, without further restriction, that

$$\lim \int_a^b \frac{\Delta f}{\Delta y} = \int_a^b \lim \frac{\Delta f}{\Delta y}. \tag{5}$$

As

$$\lim \frac{\Delta f}{\Delta y} = \frac{\partial f}{\partial y},$$

4) and 5) give

$$\frac{dJ}{dy} = \int_a^b \frac{\partial f}{\partial y}\, dx.$$

But we have already seen in 562, 2, that an interchange of the symbols

$$\lim, \qquad \int$$

is not always permissible.

4. *Example.*

$$J = \int_0^\pi \log(1 - 2y\cos x + y^2)dx. \tag{6}$$

Here, $f(x, y)$ and

$$f_y'(xy) = \frac{2(y - \cos x)}{1 - 2y\cos x + y^2}$$

are continuous in the rectangle $(0, \pi, \alpha, \beta)$ if (α, β) does not contain the points $y = \pm 1$. Cf. 560, Ex. 2. Hence

$$\frac{dJ}{dy} = 2\int_0^\pi \frac{(y - \cos x)dx}{1 - 2y\cos x + y^2}, \qquad |y| \neq 1. \tag{7}$$

5. The following is an example where the conditions of theorem 1 are not satisfied, and where differentiation under the integral sign leads to a wrong result.

Let

$$F(x, y) = \frac{xy}{\sqrt{x^2 + y^2}}, \qquad \text{except at origin;}$$

$$= 0, \qquad \text{at origin.}$$

Then

$$D_x F(x, y) = \frac{y^3}{(x^2 + y^2)^{\frac{3}{2}}}, \qquad \text{except at origin;}$$

$$= 0, \qquad \text{at origin.}$$

Hence

$$\int_0^1 D_x F(xy)dx = \frac{y}{\sqrt{1 + y^2}}, \qquad y \text{ arbitrary.}$$

Let

$$f(x, y) = \frac{y^3}{(x^2 + y^2)^{\frac{3}{2}}}, \qquad \text{except at origin;}$$

$$= 1, \qquad \text{at origin.}$$

Then

$$J(y) = \int_0^1 f dx = \int_0^1 D_x F dx = \frac{y}{\sqrt{1+y^2}}.$$

Hence

$$J'(0) = 1. \tag{8}$$

On the other hand,

$$f_y'(x, 0) = 0, \qquad x \text{ arbitrary.}$$

Hence

$$\int_0^1 f_y'(x, 0) dx = 0. \tag{9}$$

The equations 8), 9) show that in this case differentiation under the integral sign is not permissible. In fact, we observe here that

$$f_y'(x, y) = \frac{3x^2y^2}{(x^2+y^2)^{\frac{5}{2}}}, \qquad \text{except at origin;}$$

$$= 0, \qquad \text{at origin,}$$

and is therefore not limited about the origin. Thus, condition 1° of theorem **1** is not fulfilled.

Integration

568. 1. Let $f(xy)$ be continuous in the rectangle $R = (a, b, \alpha, \beta)$. Since

$$\int_a^x f dx \qquad a \leq x \leq b. \tag{1}$$

is a continuous function of y by 563, 2, it is integrable in (α, β). Therefore

$$\int_\alpha^y dy \int_a^x f dx \tag{2}$$

is convergent for each $\alpha \leq y \leq \beta$.

The integral 2) is obtained from 1) by *integrating with respect to the parameter* y. It is called a *double iterated integral;* or more shortly, when no ambiguity can arise, a *double integral.*

2. *Let $f(x, y)$ be continuous in $R = (ab\alpha\beta)$. Let*

$$F(x, y) = \int_\alpha^y dy \int_a^x f dx, \qquad x, y \text{ in } R.$$

Then

$$F''_{xy} = F''_{yx} = f(xy), \qquad \text{in } R. \tag{3}$$

For, by 537,

$$F'_y = \int_a^x f dx,$$

$$F''_{yx} = f(xy). \tag{4}$$

On the other hand, by 567, 2,

$$F'_x = \int_a^y dy \frac{\partial}{\partial x} \int_a^x f dx$$

$$= \int_a^y f dy, \quad \text{by 537.}$$

Hence, by 537,

$$F''_{xy} = f(xy). \tag{5}$$

The relation 3) follows now from 4), 5).

Inversion of the Order of Integration

569. The integral of 568, viz.

$$\int_a^y dy \int_a^x f dx, \tag{1}$$

is obtained by integrating first with respect to x, and then with respect to y.

But we might have integrated in the inverse order, getting

$$\int_a^x dx \int_a^y f dy. \tag{2}$$

It frequently happens that the integrals 1), 2) are equal, and this fact is of greatest importance in transforming such integrals. When these two integrals are the same, we say that we can *invert the order of integration*, or the integral 2) *admits inversion*. We give now a simple case when this inversion is possible. Later we shall give a broader criterion.

570. 1. *Let $f(xy)$ be limited in the rectangle $R = (ab\alpha\beta)$. Let it be continuous in R, except possibly along a finite number of lines parallel to the x or y axes, along which, however, f is integrable. Then*

$$\int_\alpha^\beta dy \int_a^b f(xy) dx = \int_a^b dx \int_\alpha^\beta f(xy) dy. \tag{1}$$

Case 1. Let f be continuous in R without exception.

We saw in 568 that

$$\frac{\partial}{\partial x}\cdot\frac{\partial F}{\partial y}=f(xy).$$

Hence, by 538,

$$\int_a^b f dx=\frac{\partial F(b,\,y)}{\partial y}-\frac{\partial F(a,\,y)}{\partial y}.$$

For the same reason,

$$\int_\alpha^\beta dy\int_a^b f dx=\int_\alpha^\beta\frac{\partial F(b,\,y)}{\partial y}dy-\int_\alpha^\beta\frac{\partial F(a,\,y)}{\partial y}dy$$

$$=F(b,\,\beta)-F(b,\,\alpha)-F(a,\,\beta)+F(a,\,\alpha). \quad (2$$

Similarly, $$\int_\alpha^\beta f dy=\frac{\partial F(x\beta)}{\partial x}-\frac{\partial F(x\alpha)}{\partial x},$$

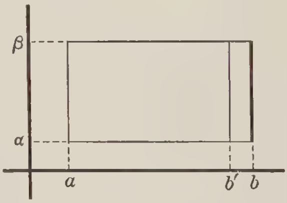
and $$\int_a^b dx\int_\alpha^\beta f dy=F(b\beta)-F(a\beta)-F(b\alpha)+F(a\alpha). \quad (3$$

From 2), 3) we get 1).

Case 2. Let f be continuous in R, except for points on the line $x=b$.

By 1 we have, $a<b'<b$,

$$\int_a^{b'}dx\int_\alpha^\beta f dy=\int_\alpha^\beta dy\int_a^{b'}f dx. \quad (4$$



Moreover, by 563, 3,

$$\int_\alpha^\beta f(x,\,y)dy$$

being a continuous function of x in $(a,\,b)$ except possibly at b, and limited in $(a,\,b)$,

$$\int_a^b dx\int_\alpha^\beta f dy$$

exists.

Now, by 536,

$$\lim_{b'=b}\int_a^{b'}dx\int_\alpha^\beta f dy=\int_a^b dx\int_\alpha^\beta f dy. \quad (5$$

On the other hand,

$$\int_a^{b'} f dx = \int_a^b f dx - \int_{b'}^b f dx.$$

But since f is limited, let

$$|f| \leq M, \qquad \text{in } R.$$

Then

$$\left| \int_{b'}^b f dx \right| \leq M(b - b').$$

Hence the right side of 4) gives

$$\int_\alpha^\beta dy \int_a^{b'} f dx = \int_\alpha^\beta dy \int_a^b f dx - \int_\alpha^\beta dy \int_{b'}^b f dx.$$

Thus

$$\left| \int_a^{b'} dx \int_\alpha^\beta f dy - \int_\alpha^\beta dy \int_a^b f dx \right| \leq M(\beta - \alpha)(b - b').$$

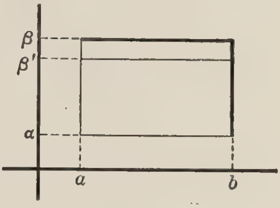
Letting $b' \doteq b$ and using 5), we get 1).

Evidently the same reasoning applies when f is continuous in R, except on one of the sides of R parallel to x-axis.

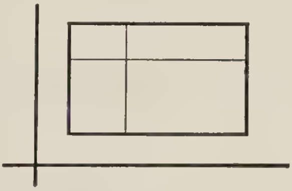
Case 3. Let f be continuous in R except on the two lines, $x = b$, $y = \beta$.

Let $\alpha < \beta' < \beta$. Then, by Case 2,

$$\int_\alpha^{\beta'} dy \int_a^b f dx = \int_a^b dx \int_\alpha^{\beta'} f dy.$$



We can reason with this equation in the same manner as we did with 4). This proves the theorem also for this case.



Case 4. If $f(xy)$ is discontinuous within R, we have only to divide R into four rectangles $\mathfrak{R}$, as in the figure.

Then each of the rectangles $\mathfrak{R}$ falls under Case 3. By breaking up the given integrals into these rectangles $\mathfrak{R}$, we prove 1) readily.

Case 5. *General Case.* By breaking R into smaller rectangles $\mathfrak{R}$, bounded by the lines on which the points of discontinuity lie, we reduce this case to Case 4.

2. We have shown in 1 that inversion is permissible when $f(xy)$ is continuous, except at points lying on a finite number of lines parallel to the x and y axes. It will be shown in Chapter XVI that inversion is permissible under much wider circumstances.

The points of discontinuity may lie on an infinite number of lines; moreover, these lines do not need to be parallel to the axes; they do not even need to be right lines.

Example. We saw by 538, 2, that

$$\int_0^{\pi/2} \frac{dx}{1 + y^2 \tan^2 x} = \frac{\pi}{2} \cdot \frac{1}{1 + |y|}.$$

Hence
$$J = \int_0^1 dy \int_0^{\pi/2} \frac{dx}{1 + y^2 \tan^2 x} = \frac{\pi}{2} \int_0^1 \frac{dy}{1 + y} = \frac{\pi}{2} \log 2. \tag{1}$$

As the integrand is limited in the rectangle $\left(0 \frac{\pi}{2} 01\right)$, and is discontinuous only when $x = \pi/2$, we can invert the order of integration. Hence

$$J = \int_0^{\pi/2} dx \int_0^1 \frac{dy}{1 + y^2 \tan^2 x} = \int_0^{\pi/2} dx \left[\frac{\text{arc tg}\,(y \tan x)}{\tan x}\right]_0^1$$
$$= \int_0^{\pi/2} \frac{x\,dx}{\tan x}. \tag{2}$$

Thus 1), 2) give
$$\int_0^{\pi/2} \frac{x\,dx}{\tan x} = \frac{\pi}{2} \log 2.$$

CHAPTER XIV

IMPROPER INTEGRALS. INTEGRAND INFINITE

Preliminary Definitions

571. 1. Up to the present, we have considered only integrals

$$\int_a^b f(x)dx,$$

in which the integrand is limited, as well as the interval of integration $\mathfrak{A} \doteq (a, b)$.

It is desirable to extend the definition of an integral to embrace integrands and intervals of integration which are not limited. Such integrals, we said, are called improper integrals.

In this chapter we consider improper integrals for which $\mathfrak{A}$ is limited, and $f(x)$ is unlimited in $\mathfrak{A}$.

$$J = \int_0^x \frac{dx}{x}, \qquad K = \int_0^1 \frac{dx}{\sqrt{1-x^2}},$$

are examples of such integrals.

2. The reader will observe that the integrand of J is not defined at $x = 0$, and the integrand of K at $x = 1$. When convenient, we may assign to the integrand at such points any value at pleasure. Cf. 598.

572. Let $f(x)$ have a finite number of points of infinite discontinuity in $\mathfrak{A}$, 347,

$$c_1, c_2, \cdots c_m.$$

We shall call these *singular points*, and say $f(x)$ is *in general limited in* $\mathfrak{A}$; or that it is *limited except at these points.* Let us inclose each point c_κ *within* a little interval $\mathfrak{C}_\kappa$ containing no other singular points. Let $\mathfrak{B}$ be what is left after removing the intervals $\mathfrak{C}_\kappa$ from $\mathfrak{A}$. On varying the intervals $\mathfrak{C}_\kappa$, $\mathfrak{B}$ will vary. If for each choice of $\mathfrak{B}$, $f(x)$ is integrable in $\mathfrak{B}$, we say $f(x)$ is *regular in* $\mathfrak{A}$ *except at the points* $c_1 \cdots c_m$; or we say $f(x)$ is *in general regular in* $\mathfrak{A}$.

573. 1. Let $f(x)$ be regular in $\mathfrak{A} = (a, b)$ except at b. If

$$\lim_{\beta=b} \int_a^\beta f(x)dx, \qquad a < \beta < b, \tag{1}$$

is finite; we say $f(x)$ *is integrable in* $\mathfrak{A}$, and define the symbol

$$J = \int_a^b f(x)dx$$

to be this limit.

Similarly, if $f(x)$ is regular in $\mathfrak{A}$ except at a, and

$$\lim_{\alpha=a} \int_\alpha^b f(x)dx, \qquad a < \alpha < b, \tag{2}$$

is finite; we say $f(x)$ *is integrable in* $\mathfrak{A}$, and define J to be this limit.

Finally, if $f(x)$ is regular in $\mathfrak{A}$, except at a, b, and

$$\lim_{\substack{\alpha=a \\ \beta=b}} \int_\alpha^\beta f(x)dx, \qquad a < \alpha < \beta < b, \tag{3}$$

is finite; we say $f(x)$ *is integrable in* $\mathfrak{A}$, and define J to be this limit.

When these limits 1), 2), 3) are finite, we say J is *finite* or *convergent*. When these limits are infinite, we say J is *infinite*. If these limits do not exist, finite or infinite, we say J *does not exist*.

2. In

$$J = \int_a^b f dx$$

we have taken $a < b$. Precisely similar definitions would apply if $a > b$.

Obviously,

$$\int_b^a f dx = -\int_a^b f dx. \tag{4}$$

For, to fix the ideas, suppose $f(x)$ regular except at a, and let the integral on the right be convergent. Then, if $a < \alpha < b$,

$$\int_\alpha^b f dx = -\int_b^\alpha f dx, \qquad \text{by 524, 1).}$$

Passing to the limit, we get 4).

574. 1. Let $f(x)$ be regular in $\mathfrak{A} = (a, b)$, except at b.

For $f(x)$ to be integrable in $\mathfrak{A}$, according to the definition just given, it is necessary and sufficient that

$$\epsilon > 0, \qquad \delta > 0, \qquad \left| \int_a^{\beta'} f dx - \int_a^{\beta} f dx \right| < \epsilon$$

for any pair of numbers β, β' within $(b - \delta, b)$, by 284.

Since

$$\int_a^{\beta'} - \int_a^{\beta} = \int_\beta^{\beta'},$$

it is necessary and sufficient that

$$\left| \int_\beta^{\beta'} f(x) dx \right| < \epsilon. \tag{1}$$

The integral

$$\int_\beta^{\beta'} f(x) dx, \qquad b - \delta < \beta, \beta' < b, \tag{2}$$

is called the *left-hand singular integral of norm* δ *for the point* b.

Similarly,

$$\int_a^{\alpha'} f dx, \qquad a < \alpha, \alpha' < a + \delta,$$

is called the *right-hand singular integral of norm* δ *for the point* a.

We have thus this result:

Let $f(x)$ *be regular in* $\mathfrak{A} = (a, b)$ *except at an end point, say* b. *For* $f(x)$ *to be integrable in* $\mathfrak{A}$, *it is necessary and sufficient that the singular integral at* b *have the limit* 0 ; *i.e.*

$$\lim_{\delta=0} \int_\beta^{\beta'} f(x) dx = 0, \qquad a < b - \delta < \beta, \beta' < b.$$

2. When a singular integral such as 2) converges to zero as its norm $\delta \doteq 0$, we shall say the singular integral is *evanescent*.

3. *Let* $f(x)$ *be regular in* $\mathfrak{A} = (a, b)$ *except at the end points* a, b. *For*

$$J = \int_a^b f(x)dx$$

to be convergent, it is necessary and sufficient that the singular integrals

$$\int_{a'}^{a''} f dx, \qquad \int_{b'}^{b''} f dx, \qquad a < a' < a'', \qquad b' < b'' < b$$

be evanescent.

It is necessary. For, when J is convergent,

$$\epsilon > 0, \qquad \delta > 0, \qquad \left| J - \int_\alpha^\beta \right| < \frac{\epsilon}{2}, \qquad a < \alpha < a + \delta, \qquad b - \delta < \beta < b.$$

Also
$$\left| J - \int_{\alpha'}^\beta \right| < \frac{\epsilon}{2}, \qquad a < \alpha' < a + \delta.$$

Adding,
$$\left| \int_{\alpha'}^{\alpha} \right| < \epsilon, \qquad a < \alpha, \alpha' < a + \delta.$$

Thus the singular integral at a is evanescent. The same is true for b.

It is sufficient. For the singular integrals being evanescent, we have

$$\epsilon > 0, \qquad \delta > 0, \qquad \left| \int_{a'}^{a''} \right| < \frac{\epsilon}{2}, \qquad \left| \int_{b'}^{b''} \right| < \frac{\epsilon}{2}.$$

Hence
$$\left| \int_{a'}^{b'} - \int_{a''}^{b''} \right| \leq \left| \int_{a'}^{a''} \right| + \left| \int_{b'}^{b''} \right| < \epsilon.$$

Hence
$$\lim_{\alpha = a, \beta = b} \int_\alpha^\beta f dx, \qquad a < \alpha < \beta < b$$

is finite, and hence by definition J is convergent.

575. Ex. 1.
$$J = \int_0^1 \frac{dx}{\sqrt{1 - x^2}}.$$

Now, for $0 < \beta < 1$,
$$F(x) = \int_0^\beta \frac{dx}{\sqrt{1 - x^2}} = \text{arc} \sin \beta.$$

As

$$\lim_{\beta=1} \text{arc} \sin \beta = \pi/2,$$

$J = \pi/2.$

The singular integral at 1 is

$$\int_{\beta}^{\beta} \frac{dx}{\sqrt{1-x^2}} = \text{arc} \sin \beta' - \text{arc} \sin \beta. \qquad (1$$

As

$$\lim_{x=1} \text{arc} \sin x = \pi/2,$$

the difference on the right of 1) is numerically $<\epsilon$ in $V_\delta(1)$, for a sufficiently small δ.

Ex. 2.

$$J = \int_0^1 \frac{dx}{x}.$$

Here, for $0 < \alpha < 1$,

$$\int_\alpha^1 \frac{dx}{x} = -\log \alpha.$$

But

$$R \lim_{\alpha=0} \log \alpha = -\infty.$$

Hence J is infinite, viz. $J = +\infty$.

The singular integral at 0 is

$$\int_{\alpha'}^{\alpha''} \frac{dx}{x} = \log \frac{\alpha''}{\alpha'}, \qquad 0 < \alpha' < \alpha'' < \delta.$$

But

$$\lim_{\delta=0} \log \frac{\alpha''}{\alpha'}$$

does not exist, by 321.

Ex. 3.

$$J = \int_0^1 x^{\lambda-1} dx, \qquad \lambda > 0.$$

Now, if $0 < \alpha < 1$,

$$\int_\alpha^1 x^{\lambda-1} dx = \frac{1-\alpha^\lambda}{\lambda}.$$

But

$$\lim_{\alpha=0} \frac{1-\alpha^\lambda}{\lambda} = \frac{1}{\lambda}, \qquad \text{by 299, 2.}$$

Hence

$$J = \frac{1}{\lambda}.$$

576. *Let $f(x)$ be integrable in $\mathfrak{A} = (a, b)$, and regular except at a, b. Let c be any point within $\mathfrak{A}$. Then*

$$J = \int_a^b f dx = \int_a^c f dx + \int_c^b f dx. \qquad (1$$

For, since J is convergent,

$$\epsilon > 0, \quad \delta > 0, \quad \left| J - \int_\alpha^\beta \right| < \frac{\epsilon}{2}, \quad a < \alpha < a + \delta; \quad b - \delta < \beta < b. \qquad (2$$

By 574, 3, the integrals

$$\int_a^c f dx, \qquad \int_c^b f dx$$

are convergent. We can therefore take δ such that also

$$\left|\int_a^c - \int_\alpha^c\right| < \frac{\epsilon}{4}, \qquad \left|\int_c^b - \int_c^\beta\right| < \frac{\epsilon}{4}.$$

Adding these last inequalities gives

$$\left|\left(\int_a^c + \int_c^b\right) - \int_\alpha^\beta\right| < \frac{\epsilon}{2}. \tag{3}$$

Adding 2), 3) gives

$$\left|J - \left(\int_a^c + \int_c^b\right)\right| < \epsilon.$$

From this follows 1).

577. 1. *Definition.* We can now generalize as follows. Let $f(x)$ be regular in $\mathfrak{A}$ except at the singular points

$$a \leq c_1 < c_2 < \cdots < c_m \leqq b.$$

If $f(x)$ is integrable in

$$(a, c_1), \quad (c_1, c_2), \quad \cdots \quad (c_m, b),$$

we say $f(x)$ is *integrable in* $\mathfrak{A}$, and set

$$\int_a^b f dx = \int_a^{c_1} f dx + \int_{c_1}^{c_2} f dx + \cdots + \int_{c_m}^b f dx.$$

2. We can therefore say:

Let $f(x)$ be regular in $\mathfrak{A} = (a, b)$, except at certain points $c_1, c_2, \cdots c_m$. For $f(x)$ to be integrable in $\mathfrak{A}$, it is necessary and sufficient that the singular integrals at $c_1, c_2, \cdots$ be evanescent.

3. From this follows at once

If $f(x)$ is in general regular and is integrable in $\mathfrak{A}$, it is integrable in any partial interval of $\mathfrak{A}$.

4. To avoid confusion and errors of reasoning, the reader should remember that, when $f(x)$ is not limited in $\mathfrak{A}$ but yet *integrable* in $\mathfrak{A}$, there are only a finite number of singular points in $\mathfrak{A}$; and f is limited and integrable in any partial interval of $\mathfrak{A}$, not embracing one of the singular points.

Criteria for Convergence

578. 1. The integral

$$K = \int_a^b |f(x)|dx$$

is called the *adjoint integral* of

$$J = \int_a^b f(x)dx.$$

We write

$$K = Adj J.$$

Let $f(x)$ be in general regular in $\mathfrak{A}$. If the adjoint of

$$J = \int_{\mathfrak{A}} f dx$$

is convergent, J is convergent.

Let c be a singular point of $f(x)$. We wish to show that the singular integrals at this point are evanescent. To fix the ideas let us consider the left-hand singular integral. By 528,

$$\left|\int_\gamma^{\gamma'} f dx\right| \le \int_\gamma^{\gamma'} |f|\,dx, \qquad c-\delta<\gamma<\gamma'<c.$$

By hypothesis, the integral on the right vanishes in the limit $\delta = 0$. Hence the integral on the left, which is the left-hand singular integral at c, is evanescent.

When $Adj J$ is convergent in $\mathfrak{A}$, J is said to be *absolutely convergent in* $\mathfrak{A}$ and $f(x)$ is *absolutely integrable in* $\mathfrak{A}$.

2. *Let $f(x)$ be absolutely integrable in $\mathfrak{A}$, and in general, limited. Then f is absolutely integrable in any partial interval of $\mathfrak{A}$.*

The demonstration is obvious.

3. The reader should note that $f(x)$ may be integrable in $\mathfrak{A}$ and yet not absolutely integrable, as the following example shows.

Let us divide the interval $\mathfrak{A} = (0, 1)$ into partial intervals, by inserting the points, $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots$

In the interval $\mathfrak{A}_n = \left(\frac{1}{n+1}, \frac{1}{n}\right)$, $n = 1, 2, \cdots$ let us erect a rectangle R_n of area $\frac{1}{n}$. Let these rectangles lie alternately above and below the x-axis. In the interval $\mathfrak{A}_n$, excluding the left-hand end point, let $f(x) =$ height of R_n taken positively or negatively accordingly as R_n is above or below the axis. Then $f(x)$ has a singular point at $x = 0$.

We have now,

$$\int_0^1 f dx = \lim_{a=0} \int_a^1 f dx = \lim_{m=\infty} \int_{\frac{1}{m}}^1 f dx = \lim_{m=\infty} \left(1 - \frac{1}{2} + \frac{1}{3} - \cdots + \frac{(-1)^m}{m-1}\right).$$

Also,

$$\int_0^1 |f|\, dx = \lim_{m=\infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{m-1}\right).$$

As the reader probably knows, or as will be shown later, the first limit is finite, the second infinite. Thus f is integrable but not absolutely integrable in $\mathfrak{A}$.

4. The reader should note this difference between proper and improper integrals,

$$J = \int_a^b f(x) dx.$$

If J is an *improper* integral, we have just seen that J is convergent if

$$K = \int_a^b |f(x)|\, dx$$

is convergent.

But if J is a *proper* integral, we saw in 528, 2, that we could not conclude the existence of J from that of K.

On the other hand, if J is a proper integral, the existence of K follows from that of J, by 507; while if J is an improper integral, we cannot conclude the convergence of K from that of J, by 3.

579. 1. *The μ test. Let $f(x)$ be regular in $\mathfrak{A} = (a, b)$ except at a. For some $0 < \mu < 1$, and $M > 0$, let there exist a $V^*(a)$ such that*

$$(x-a)^\mu |f(x)| < M, \text{ in } V^*.$$

Then $f(x)$ is absolutely integrable in $\mathfrak{A}$.

Consider the singular integral at a. We have

$$0 \le \int_{a'}^{a''} |f|\, dx \le \int \frac{M dx}{(x-a)^\mu} = \frac{M}{1-\mu} \left\{ (a''-a)^{1-\mu} - (a'-a)^{1-\mu} \right\},$$

which vanishes in the limit, by 299, 2.

2. As a corollary we have:

Let $f(x)$ be regular in $\mathfrak{A} = (a, b)$ except at a.

For some $0 < \mu < 1$, let

$$R \lim_{x=a} (x-a)^\mu |f(x)|$$

be finite. Then $f(x)$ is absolutely integrable in $\mathfrak{A}$.

Ex. 1.

$$\int_0^1 \log x \, dx$$

is convergent.

For, by 454, Ex. 2,

$$R \lim_{x=0} x^\mu |\log x| = 0, \qquad \mu > 0.$$

Ex. 2.

$$\int_0^1 x^{-\frac{1}{3}} \sin \frac{1}{x} \, dx$$

is convergent.

For,

$$R \lim_{x=0} x^\mu \cdot \left| x^{-\frac{1}{3}} \sin \frac{1}{x} \right| = 0, \qquad \mu > \tfrac{1}{3}.$$

580. *Let $f(x)$ be regular in $\mathfrak{A} = (a, b)$ except at a. In $V^*(a)$ let $f(x)$ have one sign σ, while*

$$(x-a)\sigma f(x) > M > 0.$$

Then

$$J = \int_a^b f dx = \sigma \infty.$$

For, let

$$a < \alpha < c < a + \delta.$$

Then

$$\left| \int_\alpha^c f dx \right| \geq \int_\alpha^c \frac{M}{x-a} dx = M \log \frac{c-a}{\alpha - a}$$

$$\doteq + \infty, \text{ when } \alpha \doteq a.$$

Hence

$$\int_a^c f dx = \sigma \cdot \infty.$$

Example.

$$J = \int_0^1 \frac{dx}{1-x^2} = +\infty.$$

For,

$$(1-x)f(x) = \frac{1}{1+x} > \frac{1}{2},$$

for x near 1.

581. *Let $f(x)$ be regular in $\mathfrak{A} = (a, b)$, except at a. Let*

$$\lambda = \lim_{x=a} (x - a) f(x)$$

exist. If $f(x)$ is integrable, λ must be 0.

Let us prove the theorem by showing that the contrary leads to a contradiction.

To fix the ideas suppose $\lambda > 0$. Then, for each μ such that

$$0 < \mu < \lambda$$

there exists a $V_\delta^*(a)$ such that

$$(x - a) f(x) > \mu, \qquad \text{in } V_\delta^*.$$

Then the singular integral,

$$\int_a^c f dx = + \infty, \quad \text{by 580.}$$

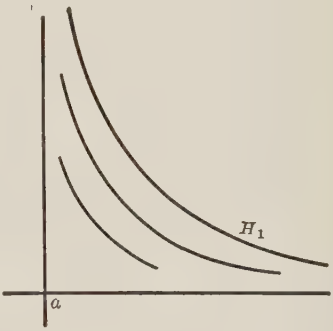
Hence f is not integrable in $\mathfrak{A}$.

582. The criteria of 579, 580 admit a simple geometric interpretation.

Consider the family of curves

$$H_\mu; \qquad (x - a)^\mu y = M, \qquad M > 0, \quad \mu > 0,$$

in the vicinity $RV^*(a)$.



The curve H_1 is a hyperbola.

If $\mu < 1$, H_μ lies below H_1, while if $\mu > 1$, H_μ lies above H_1. Furthermore, if $1 > \mu > \mu'$, H_μ lies above $H_{\mu'}$. The curves H_μ all cut each other at the point $x = a + 1$. As we are only interested in these curves in the immediate vicinity of the point $x = a$, the point $a + 1$ lies beyond the range of the figure.

The μ tests may now be stated as follows:

If in some $V^*(a)$, $|f(x)|$ remains below some H_μ which lies below H_1, $f(x)$ is integrable. If, on the other hand, $f(x)$ has one sign in $V^*(a)$, and $|f(x)|$ remains above H_1, the corresponding integral is infinite.

583. Ex. 1.

$$J=\int_0^1\frac{dx}{\sqrt{1-x^2}}.$$

The only singular point is $x=1$. Let us apply the μ test at this point. Since

$$f(x)=\frac{1}{\sqrt{1-x^2}}=\frac{1}{\sqrt{1-x}}\cdot\frac{1}{\sqrt{1+x}},$$

we see that

$$L\lim_{x=1}(1-x)^{\frac{1}{2}}f(x)=\frac{1}{\sqrt{2}}.$$

We may therefore take $\mu=\frac{1}{2}$, and J is convergent.

584. Ex. 2.

$$J=\int_1^{\frac{1}{\kappa}}\frac{dx}{\sqrt{x^2-1\cdot 1-\kappa^2x^2}},\qquad 0<\kappa<1.$$

The singular points are 1, $\frac{1}{\kappa}$.

Consider the point $x=1$.

$$\sqrt{x^2-1\cdot 1-\kappa^2x^2}=\sqrt{x-1}\,\sqrt{x+1\cdot 1-\kappa^2x^2}.$$

Hence

$$R\lim_{x=1}(x-1)^{\frac{1}{2}}f(x)=\frac{1}{\sqrt{2(1-\kappa^2)}}.$$

In the μ test we can therefore take $\mu=\frac{1}{2}$.

Consider the point $x=\frac{1}{\kappa}$.

As

$$L\lim_{x=\frac{1}{\kappa}}\left(\frac{1}{\kappa}-x\right)^{\frac{1}{2}}f(x)=\frac{\kappa^{\frac{1}{2}}}{\sqrt{2(1-\kappa^2)}},$$

we can take $\mu=\frac{1}{2}$ at this point.

Thus J is convergent.

585. Ex. 3.

$$J=\int_0^x\log\sin x\,dx,\qquad 0<x<\pi.$$

The singular point is $x=0$.

We saw

$$\lim_{x=0}\frac{\sin x}{x}=1.$$

Hence

$$\sin x=xg(x),$$

where

$$\lim_{x=0}g(x)=1.$$

Thus

$$\log\sin x=\log x+\log g(x),$$

and

$$x^\mu\log\sin x=x^\mu\log x+x^\mu\log g(x),\qquad \mu>0.$$

But

$$\lim_{x=0}x^\mu\log x=0,\qquad \lim_{x=0}x^\mu\log g(x)=0.$$

Hence

$$\lim x^\mu\log\sin x=0.$$

Thus in the μ test, we can take for μ any positive number <1. Hence J is convergent.

586. Ex. 4.
$$J = \int_0^a \frac{\cos x}{x^\mu} dx, \qquad a > 0.$$

The singular point of the integrand $f(x)$ is $x = 0$. In its vicinity $V^*(0)$, $f(x)$ has one sign $\sigma = +1$.

Then, by 579, J is convergent for $\mu < 1$; and, by 580, it is divergent for $\mu \geqq 1$.

587 Ex. 5.
$$J = \int_0^a \frac{\sin x}{x^\mu} dx, \qquad a > 0.$$

The only singular point is $x = 0$. In $V^*(0)$, the integrand has one sign $\sigma = +1$. As
$$\lim_{x=1} \frac{\sin x}{x} = 1,$$
we see, by 579, that J is convergent if $\mu < 2$; and, by 580, that it is divergent, if $\mu \geqq 2$.

588. *Logarithmic tests. Let $f(x)$ be regular in $\mathfrak{A} = (a, b)$, except at a. For some $M > 0$, $\lambda > 1$, s, let there exist a $V_\delta^*(a)$, such that in it*
$$(x-a) \cdot l_1 \frac{1}{x-a} \cdot l_2 \frac{1}{x-a} \cdots l_{s-1} \frac{1}{x-a} \cdot l_s^\lambda \frac{1}{x-a} \cdot |f(x)| < M. \qquad (1$$

Then f is absolutely integrable in $\mathfrak{A}$.

By 389, 5), for $x > a$ sufficiently near a, and $s = 1, 2, \cdots$
$$D_x l_s^{1-\lambda} \frac{1}{x-a} = \frac{\lambda - 1}{(x-a)\, l_1 \frac{1}{x-a}\, l_2 \frac{1}{x-a} \cdots l_s^\lambda \frac{1}{x-a}}.$$

Integrating, we have, for $a < \alpha' < \alpha'' < a + \delta$,
$$\int_{\alpha'}^{\alpha''} \frac{dx}{(x-a)\, l_1 \frac{1}{x-a} \cdots l_s^\lambda \frac{1}{x-a}} = \frac{1}{\lambda - 1}\left[l_s^{1-\lambda} \frac{1}{\alpha'' - a} - l_s^{1-\lambda} \frac{1}{\alpha' - a} \right],$$
$$< \epsilon, \qquad \text{for } \delta \text{ sufficiently small.}$$

Thus the singular integral of $|f(x)|$ at a is evanescent; for
$$\int_{\alpha'}^{\alpha''} |f(x)|\, dx \leq M\epsilon.$$

589. *Let $f(x)$ be regular in $\mathfrak{A} = (a, b)$ except at a. Let $f(x)$ have one sign σ in $V^*(a)$, where*

$$(x-a)l_1\frac{1}{x-a}\cdots l_s\frac{1}{x-a}\cdot\sigma f(x) > M > 0.$$

Then

$$J = \int_a^b f dx = \sigma \cdot \infty.$$

From 389, 4), for $s = 1, 2, \cdots$, and $x > a$ sufficiently near a,

$$D_x l_{s+1}\frac{1}{x-a} = \frac{-1}{(x-a)l_1\frac{1}{x-a}\cdots l_s\frac{1}{x-a}}.$$

Integrating,

$$\int_\alpha^c \frac{dx}{(x-a)l_1\frac{1}{x-a}\cdots l_s\frac{1}{x-a}} = l_{s+1}\frac{c-a}{\alpha-a}, \qquad a < \alpha < c < a+\delta.$$

Hence

$$\left|\int_\alpha^c f dx\right| > M l_{s+1}\frac{c-a}{\alpha-a},$$

$$\doteq +\infty, \qquad \text{when } \alpha \doteq a.$$

Hence

$$J = \sigma \cdot \infty.$$

590. The logarithmic tests 588, 589 admit a simple geometric interpretation.

Consider the family of curves

$$C_{s,\lambda}; \qquad y = \frac{C}{(x-a)l_1\frac{1}{x-a}\cdots l_s^\lambda\frac{1}{x-a}}, \qquad \lambda > 1;$$

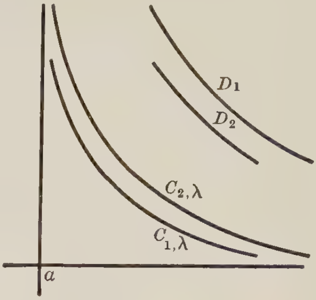
and

$$D_s; \qquad y = \frac{D}{(x-a)l_1\frac{1}{x-a}\cdots l_s\frac{1}{x-a}},$$

in $RV^*(a)$.

It is shown readily that any C curve finally lies constantly below any D curve.

For a given λ, the C curves rise as s increases; while the D curves sink as in the figure.



The logarithmic tests may now be stated as follows:

If $|f(x)|$ finally remains below some C curve, $f(x)$ is integrable. On the other hand, if $f(x)$ preserves one sign near a, and $|f(x)|$ remains above some D curve, the corresponding integral is infinite.

Properties of Improper Integrals

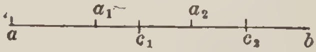
591. In the following, as heretofore in this chapter, we shall suppose that the integrands have but a finite number of singular points in the intervals considered.

When $f(x)$ has more than one singular point in $\mathfrak{A} = (a, b)$, we can break $\mathfrak{A}$ into partial intervals, such that $f(x)$ has a singular point only at one end of each such interval.

For example, if the points a, c_1, c_2 are the singular points of $f(x)$ in $\mathfrak{A}$, we have, by 576, 577, f being integrable,

$$\int_a^b f dx = \int_a^{a_1} + \int_{a_1}^{c_1} + \int_{c_1}^{a_2} + \int_{a_2}^{c_2} + \int_{c_2}^{b},$$

where a_1, a_2 are points lying between the singular points.



On account of this property, we may simplify the form of our demonstration often, by supposing $\mathfrak{A}$ to have but one singular point, which for convenience we shall take at the lower end of the interval.

592. *Let $f_1(x), \cdots f_n(x)$ be integrable in (a, b). Then*

$$\int_a^b (c_1 f_1 + \cdots + c_n f_n) dx = c_1 \int_a^b f_1 dx + \cdots + c_n \int_a^b f_n dx. \qquad (1$$

Suppose $f_1 \cdots f_n$ limited except at a.

Then, if $a < \alpha < b$,

$$\int_\alpha^b (c_1 f_1 + \cdots + c_n f_n) dx = c_1 \int_\alpha^b f_1 dx + \cdots + c_n \int_\alpha^b f_n dx.$$

Passing to the limit, we have 1).

593. *Let $f(x)$ be integrable in (a, b). Then*

$$\int_a^b f(x) dx = \int_a^c f dx + \int_c^b f dx, \qquad a < c < b.$$

If c is a singular point of f, the above relation is a matter of definition by 577.

If c is not a singular point, the demonstration follows at once from 576, 577.

594. *In $\mathfrak{A} = (a, b)$ let $f(x)$ be integrable, and*

$$f(x) \gtreqqless M.$$

Then

$$\int_a^b f(x) dx \gtreqqless M(b - a), \qquad a < b. \tag{1}$$

For, suppose a is the only singular point in $\mathfrak{A}$.

Then, if $a < \alpha < b$,

$$\int_\alpha^b f dx \gtreqqless M(b - \alpha), \quad \text{by 526, 1.}$$

Passing to the limit $\alpha = a$, we have 1).

595. *Let $f(x)$, $g(x)$ be integrable in (a, b).*
Except possibly at the singular points, let $f(x) \geq g(x)$. Then

$$\int_a^b f dx \geq \int_a^b g dx. \tag{1}$$

Suppose f, g are limited except at a.

If $a < \alpha < b$,

$$\int_\alpha^b f dx \geq \int_\alpha^b g dx, \quad \text{by 526, 2.}$$

Passing to the limit $\alpha = a$, we have 1).

596 *Let $f(x)$, $g(x)$ be integrable in (a, b).*
Except possibly at the singular points, let $f(x) \overset{=}{>} g(x)$.
At a point of continuity c of these functions, let $f(c) > g(c)$. Then

$$\int_a^b f dx > \int_a^b g dx. \tag{1}$$

Suppose f, g are limited except at a.
Let $a < \alpha < c < b$. Then, by 527, 2,

$$\int_\alpha^b f dx > \int_\alpha^b g dx;$$

by 595,

$$\int_a^\alpha f dx \geq \int_a^\alpha g dx.$$

Adding, we have 1).

597. *Let $f(x)$ be absolutely integrable in (a, b). Then $f(x)$ is integrable in (a, b), and*

$$\left|\int_a^b f dx\right| \leq \int_a^b |f(x)| dx. \tag{1}$$

In 578 we saw that $f(x)$ is integrable in (a, b).
Suppose $f(x)$ is limited, except at a.
Let $a < \alpha < b$. Then, by 528,

$$\left|\int_\alpha^b f dx\right| \leq \int_\alpha^b |f| dx. \tag{2}$$

Passing to the limit, we have 1).
Suppose $c_1, c_2, \cdots c_s$ are the singular points. Then, if $c_\kappa < a_\kappa < c_{\kappa+1}$, $\kappa = 1, 2, \cdots s-1$,

$$\int_a^b f dx = \int_a^{c_1} + \int_{c_1}^{a_1} + \cdots + \int_{c_s}^b.$$

Hence

$$\left|\int_a^b f dx\right| \leq \left|\int_a^{c_1}\right| + \left|\int_{c_1}^{a_1}\right| + \cdots$$

$$\leq \int_a^{c_1} |f| dx + \int_{c_1}^{a_1} |f| dx + \cdots, \qquad \text{by 2)},$$

$$\overset{=}{<} \int_a^b |f| dx.$$

598. *Let $f(x)$ be integrable in $\mathfrak{A} = (a, b)$.*
Let

$$J = \int_a^b f dx.$$

We may change the values of $f(x)$ over any discrete point aggregate in $\mathfrak{A}$ without altering the value of J, provided the new values of f are limited.

Suppose f limited except at a. Let $a < \alpha < b$. Then, by 530, 2,

$$\int_\alpha^b f dx = \int_\alpha^b g dx,$$

where g is the new function.

Let now $\alpha \doteq a$. The integral on the left converges to J. Hence

$$\lim_{\alpha = a} \int_\alpha^b g dx = \int_a^b g dx = J.$$

599. 1. *Let $f(x)$, $g(x)$ be absolutely integrable in $\mathfrak{A} = (a, b)$, having none of their singular points in common. Then $h = fg$ is absolutely integrable in $\mathfrak{A}$.*

To fix the ideas, let c be a singular point of f, but not of g; $a < c < b$. Then g is limited and integrable in $V_\delta(c)$.

Consider one of the singular integrals of $|h(x)|$ at c; say the right-hand one,

$$R = \int_{\gamma'}^{\gamma''} |fg| dx, \qquad c < \gamma' < \gamma'' < c + \delta.$$

Let Θ be a mean value of $|g(x)|$ in $V(c)$. Then, by 531,

$$R = \Theta \int_{\gamma'}^{\gamma''} |f| dx.$$

But $|f(x)|$ being integrable,

$$\lim_{\delta = 0} \int_{\gamma'}^{\gamma''} |f| dx = 0.$$

Hence

$$\lim R = 0,$$

as Θ is less than some positive number M.

2. *In* $\mathfrak{A} = (a, b)$, *let* $f(x)$ *be integrable and* $g(x)$ *limited and monotone. Then* fg *is integrable in* $\mathfrak{A}$.

For simplicity, suppose f is limited except at b. We must show that

$$\epsilon > 0, \qquad \delta > 0, \qquad \left|\int_c^d fgdx\right| < \epsilon, \qquad b - \delta < c, \qquad d < b.$$

Now, by the Second Theorem of the Mean, 545,

$$\int_c^d fgdx = g(c+0)\int_c^\xi fdx + g(d-0)\int_\xi^d fdx, \qquad c \leq \xi \leq b.$$

But f being integrable in $\mathfrak{A}$, the integrals on the right are numerically as small as we choose if δ is chosen sufficiently small.

600. If $f(x)$, $g(x)$ have a singular point in common, fg may not be integrable, as the following example shows :

$$f(x) = g(x) = \frac{1}{\sqrt{1-x^2}}, \qquad \mathfrak{A} = (0, 1).$$

Here, by 583, f and g are absolutely integrable in $\mathfrak{A}$. On the other hand,

$$h = fg = \frac{1}{1-x^2}.$$

Hence, by 580, 2,

$$\int_0^1 hdx = \int_0^1 \frac{dx}{1-x^2} = +\infty,$$

and h is not integrable in (0, 1).

601. *Let* $f(x)$ *be absolutely integrable in* $\mathfrak{A} = (a, b)$. *Let* $g(x)$ *be integrable in* $\mathfrak{A}$ *and* $|g(x)| \leq G$. *Then* fg *is integrable in* $\mathfrak{A}$, *and*

$$\left|\int_{\mathfrak{A}} fgdx\right| \leq G\int_{\mathfrak{A}} |f|\, dx. \qquad (1$$

For, $g(x)$ being limited and integrable, $|g(x)|$ is also integrable, by 507. Hence $|fg|$ is integrable in $\mathfrak{A}$, by 599. For simplicity, suppose that b is the only singular point of $f(x)$. Let $a < \beta < b$. Then, by 528,

$$\left|\int_a^\beta fgdx\right| \leq \int_a^\beta |fg|\, dx$$

$$\leq G\int_a^\beta |f|\, dx, \qquad \text{by 529.}$$

Passing to the limit $\beta = b$, we get 1).

602. 1. *Let* $f(x)$ *be non-negative and integrable in* $\mathfrak{A} = (a, b)$. *Let* $g(x)$ *be integrable, and*

$$m \leq g(x) \leq M.$$

Then

$$m\int_{\mathfrak{A}} fdx \leq \int_{\mathfrak{A}} fgdx \leq M\int_{\mathfrak{A}} fdx; \tag{1}$$

or

$$\int_{\mathfrak{A}} fgdx = G\int_{\mathfrak{A}} fdx, \qquad G = Mean\, g(x). \tag{2}$$

For, as in 601, fg is integrable. If, to fix the ideas, we suppose b is the only singular point of f, we have, by 529,

$$m\int_a^\beta fdx \leq \int_a^\beta fgdx \leq M\int_a^\beta fdx, \qquad a < \beta < b,$$

which gives 1) on passing to the limit $\beta = b$.

Equation 2) is obviously only another form of 1).

2. As a corollary we have:

Let $f(x)$ *be non-negative and integrable in* (a, b); *while* $g(x)$ *is continuous.*

Then

$$\int_a^b fgdx = g(\xi)\int_a^b fdx, \qquad a \leq \xi \leq b.$$

3. By repeating the reasoning of 534 and using 596, we have:

Let $f(x)$ *be integrable and non-negative; while* $g(x)$ *is continuous in* $\mathfrak{A}$. *Let* c *be a point of continuity of* $f(x)$, *and*

$$\text{Min}\, g(x) < g(c) < \text{Max}\, g(x), \qquad \text{in } \mathfrak{A}.$$

Then

$$\int_a^b fgdx = g(\xi)\int_a^b fdx, \qquad a < \xi < b.$$

603. 1. *Let* $f(x)$ *be integrable in* $\mathfrak{A} = (a, b)$. *Then*

$$J(x) = \int_\alpha^x fdx, \qquad \alpha, x \text{ in } \mathfrak{A},$$

is a continuous function of x *in* $\mathfrak{A}$.

To fix the ideas, let $a \leq \alpha < x \leq b$.

Then

$$\Delta J = \int_x^{x+h} fdx.$$

If $f(x)$ is limited in $V(x)$,

$$\lim_{h=0} \Delta J = 0 \tag{1}$$

by 536. If x is a singular point, 1) still holds by 574, since f is integrable in $\mathfrak{A}$.

2. As corollary we have:

Let $f(x)$ be integrable in (a, b). Then

$$\lim_{x=b} \int_a^x f(x)dx = \int_a^b f(x)dx, \qquad a<x<b.$$

604. 1. *Let $f(x)$ be integrable in $\mathfrak{A}=(ab)$. If $f(x)$ is continuous at x,*

$$\frac{d}{dx}\int_\alpha^x f dx = f(x), \qquad \alpha, x \text{ in } \mathfrak{A}.$$

To fix the ideas, let $a \leq \alpha < x < b$. Since f is continuous at x, this is not a singular point of f. Let c be chosen so that $\alpha < c < x$, while (c, x) contains no singular point. Then setting

$$J=\int_\alpha^x, \quad C=\int_\alpha^c, \quad K=\int_c^x,$$

we have $J=C+K$. But, by 537,

$$\frac{dK}{dx}=f(x).$$

Hence, since C is a constant,

$$\frac{dJ}{dx}=\frac{d}{dx}(C+K)=\frac{dK}{dx}=f(x).$$

2. *Let $f(x)$ be integrable in $\mathfrak{A}=(a, b)$. If f is continuous at x,*

$$\lim_{h=0}\frac{1}{h}\int_x^{x+h} f(x)dx = f(x), \qquad x \text{ in } \mathfrak{A}.$$

This is a corollary of 1.

605. *In $\mathfrak{A}=(a, b)$ let $f(x)$ be integrable. Let it be continuous except at certain points $c_1 \cdots c_s$, where $f(x)$ may be unlimited.*

If $F(x)$ is a one-valued continuous function in $\mathfrak{A}$, having $f(x)$ as derivative, except at the points c,

$$\int_a^b f(x)dx = F(b) - F(a). \qquad (1$$

Suppose $f(x)$ is continuous except at a.
Let $a < \alpha < b$. Then, by 538,

$$\int_\alpha^b f dx = F(b) - F(\alpha). \qquad (2$$

Since F is continuous,

$$\lim_{\alpha = a} F(\alpha) = F(a).$$

Passing to the limit in 2), we get 1).

Suppose now a and c are points at which f is discontinuous. To fix the ideas, let

$$a < \alpha < c < b.$$

Then

$$\int_a^b = \int_a^\alpha + \int_\alpha^c + \int_c^b. \qquad (3$$

Now as just shown,

$$\int_a^\alpha = F(\alpha) - F(a),$$

$$\int_\alpha^c = F(c) - F(\alpha),$$

$$\int_c^b = F(b) - F(c).$$

Adding, we have 1) from 3).

Ex. 1.
$$\int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}} = [\text{arc sin } x]_{-1}^{1} = \pi.$$

Here
$$f(x) = \frac{1}{\sqrt{1-x^2}}$$

is integrable in $\mathfrak{A} = (-1, 1)$. Its points of discontinuity in $\mathfrak{A}$ are $x = \pm 1$.

$$F(x) = \text{arc sin } x$$

is one-valued and continuous in $\mathfrak{A}$, and has $f(x)$ as derivative.

Ex. 2.
$$\int_{-1}^{1} \frac{dx}{x^2} = \left[-\frac{1}{x}\right]_{-1}^{1} = -2.$$

This result is obviously *false*, since the integrand is positive. The integrand is not an integrable function in $(-1, 1)$, by 580.

Change of Variable

606. 1. *Let $f(x)$ be in general regular in $\mathfrak{A}=(a, b)$ $a \lesseqgtr b$. Let*

$$u=\phi(x)$$

have a continuous derivative $\phi'(x) \neq 0$, in $\mathfrak{A}$. Let $\mathfrak{B}=(\alpha, \beta)$ be the image of $\mathfrak{A}$, and let

$$x=\psi(u)$$

be the inverse function of ϕ. If either

$$J_x=\int_{\mathfrak{A}} f(x)dx, \text{ or } J_u=\int_{\mathfrak{B}} f[\psi(u)]\psi'(u)du$$

is convergent, the other is, and $J_x=J_u$.

By hypothesis the points of $\mathfrak{A}$, $\mathfrak{B}$ stand in 1 to 1 correspondence. To fix the ideas, let f be regular except at a. Let c, γ be corresponding points in $\mathfrak{A}$, $\mathfrak{B}$. Then, by 543,

$$\int_c^b f(x)dx=\int_\gamma^\beta f[\psi(u)]\psi'(u)du, \tag{1}$$

since $f(x)$ is limited and integrable in (c, b). If now $c \doteq a$, $\gamma \doteq \alpha$, and conversely. Thus if either integral J_x or J_u is convergent, the relation 1) shows that the other is, and both are equal.

2. *In $\mathfrak{B}=(\alpha, \beta)$, $\alpha \lesseqgtr \beta$, let $x=\psi(u)$ have a continuous derivative which may vanish over a discrete aggregate, but otherwise has one sign. Let $\mathfrak{A}=(a, b)$ be the image of $\mathfrak{B}$. Let $f(x)$ be in general regular in $\mathfrak{A}$. If either*

$$J_x=\int_{\mathfrak{A}} f(x)dx, \text{ or } J_u=\int_{\mathfrak{B}} f[\psi(u)]\psi'(u)du$$

is convergent, the other is, and $J_x=J_u$.

We employ the reasoning of 1, with the aid of 544.

607. Ex. 1.

$$J_x=\int_0^1 \frac{dx}{\sqrt{1-x^2}}.$$

Let

$$x=\sin u=\psi(u).$$

Here ψ' is positive in $\mathfrak{B}=(0, \pi/2)$ except at $\pi/2$.

Also

$$J_u = \int_0^{\pi/2} du = \frac{\pi}{2}.$$

Obviously J_u is convergent. Hence, by 606, 2,

$$J_x = \pi/2,$$

a result already obtained.

Ex. 2.

$$J_x = \int_0^1 \frac{dx}{\sqrt{(1 - x^2)(1 - \kappa^2 x^2)}}, \qquad \kappa^2 < 1.$$

Let

$$x = \sin u.$$

Then

$$J_u = \int_0^{\pi/2} \frac{du}{\sqrt{1 - \kappa^2 \sin^2 u}}.$$

But the integrand of J_u is continuous in $(0, \pi/2) = \mathfrak{B}$. Hence J_u is finite. Therefore J_x is.

Both these examples illustrate how, by a change of variable, an improper integral may be transformed into a proper integral.

Second Theorem of the Mean

608. *Let* $f(x)$ *be integrable in* $\mathfrak{A} = (a, b)$. *Let* $g(x)$ *be limited and monotone in* $\mathfrak{A}$. *Then*

$$J = \int_a^b fg dx = g(a + 0) \int_a^\xi f dx + g(b - 0) \int_\xi^b f dx, \qquad (1$$

where

$$a \leq \xi \leq b.$$

We assume that $g(a + 0)$, $g(b - 0)$ are different, as otherwise 1) is obviously true.

Let us suppose first, that f is regular in $\mathfrak{A}$ except at a. Then, by 545, 11),

$$a < \alpha < b$$

$$\int_a^b fy dx = g(\alpha + 0) \int_a^b f dx + \vartheta(\alpha) \cdot \{g(b - 0) - g(\alpha + 0)\}, \qquad (2$$

where $\vartheta(\alpha)$ is a mean value of

$$\int_x^b f dx, \qquad a \leq x \leq b. \qquad (3$$

In 2) let $\alpha \doteq a$. We have

$$\lim g(\alpha + 0) = g(a + 0),$$

$$\lim \int_\alpha^b f dx = \int_a^b f dx.$$

As all the terms in 2), except $\vartheta(\alpha)$, have a finite limit, it follows that $\vartheta(\alpha)$ must have a finite limit ϑ. Hence

$$\int_a^b fg dx = g(a+0) \int_a^b f dx + \vartheta \{ g(b-0) - g(a+0) \}. \qquad (4$$

Reasoning now as we did at the close of 545, we arrive at equation 1).

Suppose next, that f is regular in $\mathfrak{A}$, except at a, b. Then if

$$a < \beta < b,$$

$$\int_a^\beta fg dx = g(a+0) \int_a^\beta f dx + \vartheta_1(\beta) \{ g(\beta - 0) - g(a+0) \},$$

as we have just seen in 3).

Passing to the limit, we have, as before,

$$\int_a^b fg dx = g(a+0) \int_a^b f dx + \vartheta_0 \{ g(b-0) - g(a+0) \}.$$

This may be transformed as before, giving 1) also for this case.

Let us suppose finally, that the singular points of f are

$$c_1, c_2, \cdots c_s.$$

Then

$$J = \int_a^{c_1} + \int_{c_1}^{c_2} + \cdots + \int_{c_s}^b.$$

If a or b are singular points, the first or last integral may be discarded.

By the preceding,

$$\int_a^{c_1} = g(a+0) \int_a^{\xi_0} f dx + g(c_1 - 0) \int_{\xi_0}^{c_1} f dx$$

$$= g(a+0) \left\{ \int_a^b - \int_{\xi_0}^b \right\} + g(c_1 - 0) \left\{ \int_{\xi_0}^b - \int_{c_1}^b \right\}.$$

Similarly,

$$\int_{c_1}^{c_2} = g(c_1+0)\left\{\int_{c_1}^{b} - \int_{\xi_1}^{b}\right\} + g(c_2-0)\left\{\int_{\xi_1}^{b} - \int_{c_2}^{b}\right\}$$

. . . .

$$\int_{c_{s-1}}^{c_s} = g(c_{s-1}+0)\left\{\int_{c_{s-1}}^{b} - \int_{\xi_{s-1}}^{b}\right\} + g(c_s-0)\left\{\int_{\xi_{s-1}}^{b} - \int_{c_s}^{b}\right\}$$

$$\int_{c_s}^{b} = g(c_s+0)\left\{\int_{c_s}^{b} - \int_{\xi_s}^{b}\right\} + g(b-0)\int_{\xi_s}^{b}.$$

Adding all these equations, we get, setting $c_0 = a$, $c_{s+1} = b$:

$$J = g(a+0)\int_a^b + \sum_{\kappa=1}^{s}\left\{g(c_\kappa+0) - g(c_\kappa-0)\right\}\int_{c_\kappa}^{b}$$

$$+ \sum_{\kappa=0}^{s}\left\{g(c_{\kappa+1}-0) - g(c_\kappa+0)\right\}\int_{\xi_\kappa}^{b}$$

$$= g(a+0)\int_a^b + S + T. \tag{5}$$

Now $\mathfrak{m}$, $\mathfrak{M}$ denoting the extremes of the integral 3),

$$\mathfrak{m}\sum_1^s\{g(c_\kappa+0) - g(c_\kappa-0)\} \leq S \leq \mathfrak{M}\Sigma\{g(c_\kappa+0) - g(c_\kappa-0)\}$$

$$\mathfrak{m}\sum_0^s\{g(c_{\kappa+1}-0) - g(c_\kappa+0)\} \leq T \leq \mathfrak{M}\Sigma\{g(c_{\kappa+1}-0) - g(c_\kappa+0)\}.$$

Which added give

$$\mathfrak{m}(g(b-0) - g(a+0)) \leq S + T \leq \mathfrak{M}(g(b-0) - g(a+0)).$$

Thus 5) gives

$$J = g(a+0)\int_a^b + \vartheta(g(b-0) - g(a+0)).$$

This is an equation of the same form as 4). Thus reasoning as we did on 4), we get 1).

INTEGRALS DEPENDING ON A PARAMETER

Uniform Convergence

609. Let $f(x, y)$ be defined over a rectangle $R = (a, b, \alpha, \beta)$, β finite or infinite, and be unlimited in R. Let $\mathfrak{A} = (a, b)$, $\mathfrak{B} = (\alpha, \beta)$.

For each y in $\mathfrak{B}$, let

$$J(y) = \int_a^b f(x, y)dx$$

be convergent. Then J is a one-valued function of y in $\mathfrak{B}$.

As in Chapter XIII, we wish now to study J with respect to continuity, differentiation, and integration, restricting ourselves to certain simple but important cases.

610. 1. For brevity we introduce the following terms. We shall say $f(xy)$ is *regular* in $R = (ab\alpha\beta)$, β finite or infinite, when

1°. $f(xy)$ has no points of infinite discontinuity in R.

2°. $f(xy)$ is integrable in $\mathfrak{A} = (a, b)$ for each y in $\mathfrak{B} = (\alpha, \beta)$.

When β is finite, we shall sometimes need to integrate $f(xy)$ with respect to y. In this case we shall also suppose

3°. $f(xy)$ is integrable in $\mathfrak{B} = (\alpha, \beta)$ for each x in $\mathfrak{A}$.

For example, $f(x, y) = y \sin x$ is regular in R.

If β is finite, f is limited in R. If $\beta = \infty$, f is not limited in R although it has no points of infinite discontinuity.

2. If $f(xy)$ is regular in R, except that it may have points of infinite discontinuity on certain lines $x = a_1, \cdots x = a_r$, we shall say $f(xy)$ is *regular in R except on the lines $x = a_1, \cdots$; or that it is in general regular with respect to x.*

3. Let $f(xy)$ be continuous in R except on certain lines

$$x = a_1, \cdots x = a_r; \ y = \alpha_1, \cdots y = \alpha_s.$$

On the lines $x = a_1, \cdots$ it may have points of infinite discontinuity; on the lines $y = \alpha_1 \cdots$ it may have finite discontinuities. If $f(xy)$ is otherwise regular, we shall say it is *simply regular except on the lines $x = a_1, \cdots x = a_r$; $y = \alpha_1, \cdots y = \alpha_s$; or that it is simply irregular with respect to x.*

Thus the *simply* irregular functions are a special case of the functions which are in general regular.

4. The lines $x = a_1, \cdots x = a_r$, on which $f(xy)$ may have points of infinite discontinuity are called singular lines.

The integrals

$$\int_{a_\iota - \delta}^{a_\iota} f dx, \quad \int_{a_\iota}^{a_\iota + \delta} f dx \qquad \delta > 0, \text{ arbitrarily small}$$

are called the left and right hand singular integrals relative to the lines $x = a_\iota$, $\iota = 1, 2 \cdots r$.

5. In 609 we made the formal requirement that $f(xy)$ should be defined at every point of R. It usually happens in practice that f is not defined at its points of infinite discontinuity. Such is the case in such integrals as

$$\int \frac{dx}{\sqrt{xy}}, \quad \int \frac{y^2 - x^2}{(x^2 + y^2)^2} dx, \quad \int x^{y-1} \log^n x \, dx.$$

It is, however, easy to satisfy the above requirement in all the cases we shall consider; for, by 598. the value of

$$\int_a^b f(x, y) dx$$

is not affected by a change of the value of f at points lying on the lines $x = a_1 \cdots$ subject to the restrictions of that theorem.

6. This fact may also be used to advantage sometimes to simplify $f(x, y)$ by changing its value at points lying on these lines.

611. 1. Let $f(xy)$ be regular in $R = (ab\alpha\beta)$, β finite or infinite, except on $x = a_1, \cdots$

If the singular integrals relative to these lines be uniformly evanescent in $\mathfrak{B}$, we say

$$J = \int_a^b f(x, y) dx$$

is *uniformly convergent in* $\mathfrak{B}$.

2. If J is uniformly convergent in the intervals $\mathfrak{B}_1, \cdots \mathfrak{B}_m$, it is obviously uniformly convergent in their sum.

3. If J is the sum of several uniformly convergent integrals in $\mathfrak{B}$, it is itself uniformly convergent in $\mathfrak{B}$.

4. *Let $f(xy)$ be in general regular with respect to x in $R=(ab\alpha\beta)$, β finite. If J is uniformly convergent in $\mathfrak{B}$, it is limited in $\mathfrak{B}$.*

For simplicity suppose $x=b$ is the only singular line. Then

$$\left|\int_{b'}^{b} f dx\right| < \sigma, \qquad \text{uniformly in } \mathfrak{B}.$$

But in the rectangle $(ab'\alpha\beta)$,

$$|f(xy)| < M.$$

Now
$$J = \int_a^{b'} + \int_{b'}^{b}.$$

Hence
$$|J| < M(b-a) + \sigma.$$

612. *Let $f(xy)$ be regular in $R=(ab\alpha\beta)$, β finite or infinite, except on $x=b$.*

The singular integral
$$\int_{b'}^{b} f(xy)dx, \qquad b_0 \leq b' < b, \tag{1}$$

is uniformly evanescent in $\mathfrak{B}=(\alpha,\beta)$ if

$$|f(xy)| \leq \phi(x), \qquad \text{in } \mathfrak{A}'=(b_0, b),$$

and ϕ is integrable in $\mathfrak{A}'$.

For, $f(xy)$ being limited and integrable in (b', b''), where $b' < b'' < b$, we have for any y in $\mathfrak{B}$

$$\left|\int_{b'}^{b''} f dx\right| \leq \int_{b'}^{b''} |f| dx, \quad \text{by } 528$$

$$\leq \int_{b'}^{b''} \phi dx, \qquad \text{by } 526, 2.$$

But ϕ being integrable in $\mathfrak{A}'$, we can take b_0 so near b that the last integral is $<\epsilon$. But as this is independent of y,

$$\left|\int_{b'}^{b''} f(xy)dx\right| < \epsilon, \qquad \text{in } \mathfrak{B}.$$

Hence 1) is uniformly evanescent.

613. *Example.* Let us consider the integral

$$J = \int_0^\pi \log(1 - 2y\cos x + y^2)dx \tag{1}$$

for values of y in $\mathfrak{B} = (\alpha, \beta)$, β finite. The integrand $f(xy)$ is continuous except at the points

$$x = m\pi, \quad y = (-1)^m, \quad m = 0, \pm 1, \cdots$$

by 560, Ex. 2. Let us first consider the singular integral

$$S = \int_0^{a'} f dx, \qquad 0 < a' < \delta$$

relative to the line $x = 0$. Since $y = 1$ is the only point of infinite discontinuity on this line, we may restrict ourselves to an interval $\mathfrak{B}' = (1 - \sigma, 1 + \sigma)$. We set

$$y = 1 + h, \qquad |h| \leq \sigma.$$

Then

$$S = \int_0^{a'} \log \left\{ 2(1+h)\left(1 - \cos x + \frac{h^2}{2(1+h)}\right) \right\} dx$$

$$= \log 2(1+h) \int_0^{a'} dx + \int_0^{a'} \log\left(1 - \cos x + \frac{h^2}{2(1+h)}\right) dx$$

$$= S_1 + S_2.$$

Obviously S_1 is uniformly evanescent in $\mathfrak{B}'$.

To show that S_2 is uniformly evanescent, we observe that

$$\left| \log\left(1 - \cos x + \frac{h^2}{2(1+h)}\right) \right| \leq |\log(1 - \cos x)|,$$

since $\log x$ increases with x, and is negative for small values of the argument. We apply now 612. To this end we show that

$$\phi(x) = \log(1 - \cos x)$$

is absolutely integrable in $(0, a')$, using the μ-test.

Now in 454, Ex. 1, we saw that

$$R \lim_{x=0} x^\mu \log(1 - \cos x) = 0, \qquad 0 < \mu < 1.$$

Hence $|\phi|$ is integrable, and S_2 is uniformly evanescent. Hence S is uniformly evanescent not only in $\mathfrak{B}'$, but in $\mathfrak{B}$.

The same reasoning may be applied to the singular integral relative to the line $x = \pi$. Here the only point of infinite discontinuity is $y = -1$.

Hence, by 611, 2, the integral J is uniformly convergent in $\mathfrak{B}$.

614. *Example.* Let us consider the integral

$$J = \int_0^1 x^{y-1} |\log x|^n dx, \qquad n \gtreqless 0. \tag{1}$$

We show first that *it is convergent only for* $y > 0$.

For, let $y > 0$. Applying the μ-test, we have

$$\lim_{x=0} x^\mu \cdot x^{y-1} |\log x|^n = \lim x^\lambda |\log x|^n, \qquad \lambda > 0$$

$$= 0, \quad \text{by 454, Ex. 2,}$$

for properly chosen $0 < \mu < 1$.

Hence, by 579, J is convergent.

Let $y \leqq 0$. Then
$$\lim x \cdot x^{y-1} |\log x|^n = \lim x^\lambda |\log x|^n, \qquad \lambda \leqq 0$$

$$= + \infty.$$

Hence, by 580, J is divergent.

Let $0 < \alpha < \beta$. We show that J *is uniformly convergent in* $\mathfrak{B} = (\alpha, \beta)$. In the first place we note that the integrand is continuous in $R = (0, 1, \alpha, \beta)$, except on the line $x = 0$, where it has points of infinite discontinuity. We have, therefore, only to show that the singular integral S relative to this line is uniformly evanescent. To this end we use 612. Now

$$x^{y-1} |\log x|^n < x^{\alpha-1} |\log x|^n, \qquad y > \alpha.$$

But we have just seen that

$$\phi(x) = x^{\alpha-1} |\log x|^n. \tag{2}$$

is integrable. Hence S is uniformly evanescent in $\mathfrak{B}$.

615. *Let* $f(xy)$ *be regular in* $R = (ab\alpha\beta)$, β *finite or infinite, except on* $x = b$. *The singular integral*

$$\int_{b'}^b f(xy) dx, \qquad b' < b,$$

is uniformly evanescent in $\mathfrak{B} = (\alpha, \beta)$, *if*

$$f(xy) = \phi(x) g(xy), \quad \textit{in } R_b = (b - \delta, b, \alpha, \beta);$$

where 1° $\phi(x)$ *is integrable in* $\mathfrak{A}' = (b - \delta, b)$;

2° $g(xy)$ *is limited in* R_b, *and integrable in any* $(b - \delta, b'')$, $b'' < b$, *for each* y *in* $\mathfrak{B}$.

For, by 2°,
$$|g(xy)| < M.$$

Then by 1°, there exist for each $\epsilon > 0$, a $\delta > 0$ such that

$$\left| \int_{b'}^{b''} \phi(x) dx \right| < \frac{\epsilon}{M}. \tag{1}$$

Then for any y in $\mathfrak{B}$,

$$\left|\int_{b'}^{b''} f dx\right| = \left|\int_{b'}^{b''} \phi g dx\right|$$

$$< M\left|\int_{b'}^{b''} \phi dx\right|$$

$$\leq \epsilon, \text{ by } 1).$$

Continuity

616. 1. *Let $f(xy)$ be regular in $R = (ab\alpha\beta)$, β finite or infinite, except on the lines $x = a_1, \cdots x = a_r$.*

1°. *Let the singular integrals relative to these lines be uniformly evanescent in $\mathfrak{B} = (\alpha, \beta)$.*

2°. *Let η, finite or infinite, lie in $\mathfrak{B}$, and*

$$\lim_{y=\eta} f(xy) = \phi(x) \qquad \textit{uniformly}$$

in $\mathfrak{A} = (a, b)$, except possibly at $x = a_1, \cdots x = a_r$.

3°. *Let $\phi(x)$ be integrable in $\mathfrak{A}$. Then*

$$\lim_{y=\eta} \int_a^b f(xy) dx = \int_a^b \lim_{y=\eta} f(xy) dx = \int_a^b \phi(x) dx. \tag{1}$$

For simplicity, we shall suppose there is only one singular line, viz. $x = b$; we shall also take $\eta = \infty$.

Let

$$D = \int_a^b \{f(xy) - \phi(x)\} dx.$$

We wish to show that

$$\epsilon > 0, \quad G, \quad |D| < \epsilon, \qquad \textit{for any } y > G.$$

Now

$$D = \int_a^{b'} \{f(xy) - \phi(x)\} dx + \int_{b'}^b f dx - \int_{b'}^b \phi dx, \qquad b' < b,$$

$$= D_1 + D_2 + D_3.$$

Now by 1°, 3°, the last two integrals are numerically $< \epsilon/3$, if b' is sufficiently near b, for any y. On the other hand, if G is sufficiently large, we have for any $y > G$,

$$|f(xy) - \phi(x)| < \frac{\epsilon}{3(b-a)},$$

for every x in (a, b'), by virtue of 2°.

Hence $|D_1| < \epsilon/3$. Hence

$$|D| < \epsilon, \qquad y > G,$$

which establishes 1).

2. *Let $f(xy)$ be regular in $R = (ab\alpha\beta)$, β finite or infinite, except on the lines $x = a_1, \cdots x = a_r$.*

1°. *Let the singular integrals relative to these lines be uniformly evanescent in $\mathfrak{B} = (\alpha, \beta)$.*

2°. *Let η, finite or infinite, lie in $\mathfrak{B}$, and*

$$\lim_{y=\eta} f(x, y) = \phi(x) \qquad \textit{uniformly}$$

in $\mathfrak{A} = (ab)$, except possibly at $x = a_1, \cdots x = a_r$.

Then

$$j = \lim_{y=\eta} \int_a^b f(xy)dx \qquad \textit{exists.}$$

3°. *Let $\phi(x)$ be integrable in $\mathfrak{A}$. Then*

$$\lim_{y=\eta} \int_a^b f(xy)dx = \int_a^b \lim_{y=\eta} f(xy)dx = \int_a^b \phi(x)dx.$$

We need only show that j exists, since the rest follows by 1. Let us suppose, to fix the ideas, that $x = b$ is the only singular line, and that $\eta = \infty$.

Then

$$\begin{aligned} D &= \int_a^b \{f(x, y') - f(x, y'')\}dx \\ &= \int_a^{b'} \{f(x, y') - f(x, y'')\}dx + \int_{b'}^b f(x, y')dx - \int_{b'}^b f(x, y'')dx \\ &= D_1 + D_2 + D_3. \end{aligned}$$

But by 1°, there exists a b' such that

$$|D_2|, \quad |D_3| < \epsilon/4$$

for any y in $\mathfrak{B}$.

By 2°, we can take γ such that for any x in (a, b')

$$|f(xy) - \phi(x)| < \frac{\epsilon}{4(b-a)}, \qquad y > \gamma.$$

Hence

$$|f(x, y') - f(x, y'')| < \frac{\epsilon}{2(b-a)},$$

for any $y', y'' > \gamma$, and x in (a, b').

Thus

$$|D_1| < \epsilon/2.$$

Hence

$$|D| < \epsilon, \qquad \text{for any } y > \gamma;$$

and the limit j exists.

617. 1. In 561 we have defined the term $f(x, y)$ as a uniformly continuous function of y in $\mathfrak{B}$. It may happen that $f(x, y+h)$ converges to $f(xy)$ for each y in $\mathfrak{B}$ and any x in $\mathfrak{A}$, but that the uniform convergence breaks down at points lying on the lines $x = a_1, \cdots x = a_r$. In this case, we shall say f is a *regularly continuous function of y in $\mathfrak{B}$*. If $f(x, y+h)$ converges uniformly to $f(xy)$ except on $x = a_1, \cdots$, where it may not even converge to $f(x, y)$, we shall say that $f(xy)$ is a *semi-uniformly continuous function of y*.

In both cases, we can inclose the lines $x = a_1, \cdots$ in little bands of width small at pleasure but fixed, such that the convergence is uniform in $\mathfrak{B}$, when x ranges over $\mathfrak{A}$, excluding values which fall in the above bands.

2. It may happen that $f(x, y)$ is a regularly or a semi-uniformly continuous function of y in $\mathfrak{B}$ except on the lines $y = \alpha_1, \cdots y = \alpha_s$. We shall say in this case that f is *in general* regularly or semi-uniformly continuous in y.

3. We wish to make a remark here which will sometimes permit us to simplify the form of a demonstration without loss of generality. In questions of uniform convergence or uniform continuity, the uniformity may break down at points lying on certain lines $x = a_1, \cdots x = a_m$. In this case we may count such lines as

singular lines. When β is finite, and no points of infinite discontinuity lie on these lines, their singular integrals are obviously uniformly evanescent in $\mathfrak{B}$.

618. 1. As corollaries of 616 we have, using 611, 4:

Let $f(xy)$ *be in general regular with respect to* x *in* $R=(ab\alpha\beta)$, β *finite.*

Let $f(xy)$ *be a semi-uniformly continuous function of* y, *except at* $y=\alpha_1, \cdots y=\alpha_m$.

Let
$$J(y)=\int_a^b f(xy)dx \tag{1}$$

be uniformly convergent in $\mathfrak{B}=(\alpha\beta)$.
Then J *is limited in* $\mathfrak{B}$ *and continuous, except possibly at* $\alpha_1, \cdots \alpha_m$.

2. *Let* $f(xy)$ *be continuous in* R, *except on* $x=a_1, \cdots x=a_r$. *Let* 1) *be uniformly convergent in* $\mathfrak{B}$. *Then* J *is continuous in* $\mathfrak{B}$.

619. Ex. 1. The integral

$$J=\int_0^\pi \log(1-2y\cos x+y^2)dx$$

is a continuous function of y in any interval $\mathfrak{B}=(\alpha\beta)$. For the integrand is continuous in $(0, \pi, \alpha, \beta)$, except on the lines $x=0$, $x=\pi$. In 613 we saw J is uniformly convergent in $\mathfrak{B}$. Hence, by 618, 2, J is continuous.

Ex. 2. The integral
$$J=\int_0^1 x^{y-1}|\log y|^n dx$$

is continuous in (α, β). $0<\alpha<\beta$.

This follows from 618, 2, and 614.

Integration

620. 1. Up to the present we have been considering the case when the singular integrals

$$S_c=\int_{c'}^{c} f(xy)dx$$

relative to a line $x=c$ are uniformly evanescent.

For the purpose of integrating

$$J(y) = \int_a^b f(xy)dx$$

with respect to the parameter y over a finite interval $\mathfrak{B} = (\alpha, \beta)$, we can take a slightly more general case.

As before, let $f(xy)$ be in general regular in $R = (ab\alpha\beta)$. To fix the ideas, let us consider the left-hand singular integral at c.

Suppose now that for each $\epsilon > 0$ there exists a $\delta > 0$ such that for any y in $\mathfrak{B}^* = (\alpha, \beta^*)$,

$$|S_c| \leq \epsilon\sigma(y)$$

for every c' in $(c - \delta, c)$. Here σ is regular in $\mathfrak{B}$ except at β, and is integrable in $\mathfrak{B}$; it is also independent of ϵ. In this case we shall say this singular integral is *normal*. Similar remarks hold for the right-hand singular integral at c.

2. Obviously, if the singular integrals at c are normal, they are uniformly evanescent in any partial interval $(\alpha\beta')$ of $\mathfrak{B}$.

Also, if the singular integrals at c are uniformly evanescent in $\mathfrak{B}$, they are *a fortiori* normal.

3. When the singular integrals of

$$J = \int_a^b f(xy)dx$$

are normal, we shall say J is *normally convergent* in $\mathfrak{B}$.

4. *If the singular integrals at c are normal, there exists for each* $\epsilon > 0$, a $\delta > 0$, *such that*

$$\left|\int_\lambda^\mu dy \int_{c'}^c f dx\right| \leq \epsilon,$$

for any c' *in* $(c - \delta, c)$ *or* $(c, c + \delta)$, *and any* λ, μ *in* $\mathfrak{B}$.

To fix the ideas consider only the left-hand singular integral.

Then

$$\left|\int_{c'}^c f dx\right| < \frac{\epsilon\sigma(y)}{S},$$

where

$$S = \int_\alpha^\beta \sigma dy \geqq \int_\lambda^\mu \sigma dy.$$

Hence

$$\left|\int_\lambda^\mu \int_{c'}^c\right| \leq \int_\lambda^\mu \left|\int_{c'}^c\right| \leq \frac{\epsilon}{S}\int_\lambda^\mu \leq \epsilon.$$

621. *Let $f(xy)$ be in general regular with respect to x, in $R=(ab\alpha\beta)$, β finite.*

1°. *Let f be in general a semi-uniformly continuous function of y in $\mathfrak{B}=(\alpha, \beta)$.*

2°. *Let*

$$J(y)=\int_a^b f(xy)dx$$

be normally convergent in $\mathfrak{B}$.

Then J is integrable in $\mathfrak{B}$.

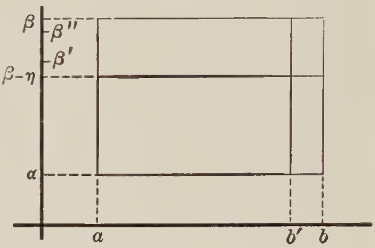
To fix the ideas let $x=b$ be the only singular line. Since the singular integral is normal in $\mathfrak{B}$, it is uniformly evanescent in any $(\alpha\gamma)$, $\gamma<\beta$, by 620, 2. Hence, by 618, 1, J is integrable in (α, γ).

To show that J is integrable in $\mathfrak{B}$, we have only to show that

$$T=\int_{\beta'}^{\beta''} Jdy,$$

$$\beta-\eta<\beta'<\beta''<\beta,$$

converges to 0 as $\eta \doteq 0$.



Now

$$T=\int_{\beta'}^{\beta''} dy\int_a^b fdx=\int_{\beta'}^{\beta''}\int_a^{b'}+\int_{\beta'}^{\beta''}\int_{b'}^{b}=T_1+T_2. \qquad (1$$

But, by 620, 4, there exists a b' such that

$$|T_2|<\epsilon/2, \qquad b'<b,$$

however β', β'' are chosen.

On the other hand, $f(xy)$ being by hypothesis limited in any

$$(a, b', \alpha, \beta), \qquad b'<b.$$

$$\left|\int_a^{b'} fdx\right|<M.$$

We can therefore choose η so small that

$$|T_1|<\epsilon/2.$$

Hence 1) shows that for each $\epsilon>0$, there exists an $\eta>0$, such that

$$|T|<\epsilon$$

for any pair of values $\beta'\beta''$ in $(\beta-\eta, \beta^*)$.

Inversion

622. 1. *Let $f(xy)$ be in general regular with respect to x, in $R=(ab\alpha\beta)$.*

1°. *Let the singular integrals be normal.*

2°. *Let $f(xy)$ be in general a semi-uniformly continuous function of y in $\mathfrak{B}$.*

3°. *Let inversion of the order of integration be permissible for any rectangle in R not embracing the singular lines.*

Then

$$K=\int_\alpha^\beta dy\int_a^b f dx, \qquad L=\int_a^b dx\int_\alpha^\beta f dy$$

exist, and are equal.

For simplicity, let us suppose $x=b$ is the only singular line.

By 1°, 2°, and 621, the integral K exists.

By 3°

$$\int_\alpha^\beta dy\int_a^{b'} f dx=\int_a^{b'} dx\int_\alpha^\beta f dy. \qquad b-\delta<b'<b. \tag{1}$$

But

$$\int_\alpha^\beta\int_a^{b'}=\int_\alpha^\beta\int_a^b-\int_\alpha^\beta\int_{b'}^b, \qquad \text{since } K \text{ exists,}$$

$$=K-\int_\alpha^\beta\int_{b'}^b. \tag{2}$$

Hence 1), 2) give

$$\left|\int_a^{b'}\int_\alpha^\beta-K\right|=\left|\int_\alpha^\beta\int_{b'}^b\right| \tag{3}$$

$$<\epsilon, \qquad \text{by 620, 4}$$

if δ is taken sufficiently small.

Now, by 603, 2,

$$\int_a^b\int_\alpha^\beta=\lim_{b'=b}\int_a^{b'}\int_\alpha^\beta.$$

Hence, by 3),

$$L=\int_a^b\int_\alpha^\beta=K.$$

2. *Let $f(xy)$ be simply irregular in $R=(ab\alpha\beta)$ with respect to x Let the singular integrals be normal. Then*

$$\int_\alpha^\beta dy\int_a^b f dx, \qquad \int_a^b dx\int_\alpha^\beta f dy$$

exist and are equal.

This is a special case of 1. For, by 617, 3, f is in general a regularly continuous function of y. Thus condition 2° is satisfied. That condition 3° is fulfilled follows from 570, 1.

623. *Example.* We saw, 575, Ex. 3, that

$$\int_0^1 x^{y-1}dx = \frac{1}{y}, \qquad y > 0.$$

Hence for $0 < \alpha < \beta$,

$$\int_\alpha^\beta dy \int_0^1 x^{y-1}dx = \int_\alpha^\beta \frac{dy}{y} = \log\frac{\beta}{\alpha}. \tag{1}$$

We can, by 622, 2, invert the order of integration, since

$$\int_0^a x^{y-1}dx, \qquad 0 < a$$

is uniformly evanescent in (α, β) by 614.

Thus 1) gives, inverting,

$$\log\frac{\beta}{\alpha} = \int_0^1 dx \int_\alpha^\beta x^{y-1}dy = \int_0^1 \frac{x^{\beta-1} - x^{\alpha-1}}{\log x}\,dx. \tag{2}$$

For $\alpha = 1$, this gives

$$\int_0^1 \frac{x^{\beta-1} - 1}{\log x}\,dx = \log\beta. \tag{3}$$

If we set here $\beta = 2$, it gives

$$\int_0^1 \frac{x - 1}{\log x}\,dx = \log 2. \tag{4}$$

624. We give now an example where it is not permitted to invert the order of integration.

We have for all points different from the origin,

$$D_x \frac{x}{x^2 + y^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2},$$

$$D_y \frac{y}{x^2 + y^2} = \frac{x^2 - y^2}{(x^2 + y^2)^2}.$$

Thus

$$A = \int_0^1 dy \int_0^1 \frac{y^2 - x^2}{(x^2 + y^2)^2}\,dx = \int_0^1 dy \left[\frac{x}{x^2 + y^2}\right]_0^1 = \int_0^1 \frac{dy}{1 + y^2} = \frac{\pi}{4},$$

$$B = \int_0^1 dx \int_0^1 \frac{y^2 - x^2}{(x^2 + y^2)^2}\,dy = -\int_0^1 dx \left[\frac{y}{x^2 + y^2}\right]_0^1 = -\int_0^1 \frac{dx}{1 + x^2} = -\frac{\pi}{4}.$$

Hence A, B are both convergent; but they are not equal.

625. 1. Up to the present we have supposed that the points of infinite discontinuity of the integrand $f(xy)$ lie on certain lines parallel to the y-axis.

We consider now a more general case.

Let us suppose that these points of infinite discontinuity do not lie only on a finite number of lines parallel to the y-axis, but that it is necessary to employ in addition a finite number of lines parallel to the x-axis.

To fix the ideas, let these lines be

$$x = a_1, \cdots x = a_r; \quad y = \alpha_1, \cdots y = \alpha_s. \tag*{(1}$$

If $f(xy)$ is otherwise regular, *i.e.* if properties 2°, 3° of 610, 1 hold, we shall say $f(xy)$ is *regular in R except on the lines* 1), or that it is *in general regular with respect to x, y.*

2. Similarly we extend the term *simply regular*, viz.: If $f(xy)$ is continuous in R except on a finite number of lines parallel to each axis, say the lines 1), where it may have points of finite or infinite discontinuity; if, moreover, it enjoys properties 2°, 3° of 610, 1, we shall say $f(xy)$ is simply regular in R except on the lines 1), or that it is simply irregular with respect to x, y.

3. The lines $y = \alpha_1, \cdots$ are also called singular lines, and the integrals

$$\int_{\alpha'_m}^{\alpha_m} f dy$$

are singular integrals relative to the lines $y = \alpha_m$, $m = 1, 2 \cdots s$.

4. In accordance with the present assumptions, we should modify the definition of normal singular integrals given in 620, 1, so as to allow $\sigma(y)$ to have singular points at $\alpha_1, \cdots \alpha_s$.

As an example, consider

$$f(xy) = \frac{1}{\sqrt{(x-b)(y-\beta)}}.$$

Here every point on the lines $x = b$, $y = \beta$ are points of infinite discontinuity.

If $R = (a, b, \alpha, \beta)$, $a < b$, $\alpha < \beta$, we see at once that f is continuous in R except on the above lines. Obviously, f is integrable with respect to x or y in R. Thus f is simply regular in R except on the lines $x = b$, $y = \beta$.

626. 1. *Let* $f(xy)$ *be regular in* $R=(ab\alpha\beta)$, *except on the lines* $x=a_1, \cdots x=a_r;\ y=\alpha_1, \cdots y=\alpha_s$.

1°. *Let it be in general a semi-uniformly continuous function of* y *in* $\mathfrak{B}=(\alpha\beta)$.

2°. *Let the singular integrals relative to the lines* $x=a_1, \cdots$ *be normal in* $\mathfrak{B}$.

3°. *Let the singular integrals relative to the lines* $y=\alpha_1, \cdots$ *be uniformly evanescent in any interval of* $\mathfrak{A}=(ab)$, *not embracing the points* $a_1, \cdots a_r$.

4°. *Let any integral*

$$\int_{\alpha'}^{\beta'} dy \int_{a'}^{b'} f dx$$

admit inversion if the rectangle $(a'b'\alpha'\beta')$ *does not embrace one of the singular lines.*

Then

$$J(y)=\int_a^b f(x, y)dx$$

is integrable in $\mathfrak{B}$.

For simplicity, let us assume that there is only one singular line $x=b$, and one line $y=\beta$.

We observe, first, that J is integrable in any $(\alpha\beta')$, $\beta'<\beta$ by 1°, 2°, and 620, 2, 621. Therefore, to show that J is integrable in $\mathfrak{B}$, it suffices to prove that, for each $\epsilon>0$ there exists an $\eta>0$, such that

$$T=\int_{\beta'}^{\beta''} J dy$$

is numerically $<\epsilon$ for any pair of numbers $\beta'\beta''$ in $\mathfrak{C}=(\beta-\eta, \beta)$. Now

$$T=\int_{\beta'}^{\beta''} dy \int_a^b f dx = \int_{\beta'}^{\beta''}\int_a^{b'} + \int_{\beta'}^{\beta''}\int_{b'}^{b} = T_1+T_2. \qquad (1$$

But by 2° and 620, 4,

$$|T_2|<\epsilon/2$$

if b' is taken sufficiently near b. Suppose b' so chosen and then fixed.

On account of 4°,

$$T_1=\int_a^{b'}\int_{\beta'}^{\beta''}.$$

By virtue of 3° we can take $\eta>0$ sufficiently small, so that

$$\left|\int_{\beta'}^{\beta''} f dy\right| < \frac{\epsilon}{2(b-a)}, \qquad \text{in } (a, b').$$

Hence

$$|T_1| < \epsilon/2.$$

Therefore

$$|T| < \epsilon, \qquad \text{for any } \beta'\beta'' \text{ in } \mathfrak{C}.$$

2. *Let $f(xy)$ be simply regular in $R = (ab\alpha\beta)$, except on*

$$x = a_1, \cdots \; y = \alpha_1, \cdots$$

1°. *Let the singular integrals relative to $x = a_1, \cdots$ be normal in $\mathfrak{B}$.*

2°. *Let the singular integrals relative to $y = \alpha_1, \cdots$ be uniformly evanescent in any interval of $\mathfrak{A}$ not embracing the points $a_1, \cdots$*

Then

$$J(y) = \int_a^b f(x, y)dx$$

is integrable in $\mathfrak{B}$.

For, conditions 1°, 2°, 3° of 1 are obviously fulfilled.

That 4° is satisfied, follows from 570.

627. 1. *Let $f(xy)$ be regular in $R = (ab\alpha\beta)$ except on the lines $x = a_1, \cdots x = a_r;\; y = \alpha_1, \cdots y = \alpha_s$.*

1°. *Let the singular integrals relative to the lines $x = a_1, \cdots$ be normal.*

2°. *Let the integral*

$$\int_c^d dx \int_\alpha^\beta f dy$$

admit inversion, if (c, d) does not embrace the points $a_1, a_2, \cdots$

3°. *Let*

$$K = \int_\alpha^\beta dy \int_a^b f dx$$

be convergent.

Then

$$L = \int_a^b dx \int_\alpha^\beta f dy$$

is convergent, and $K = L$.

For simplicity, let $x = b$, $y = \beta$ be the only singular lines. Then, by 2°,

$$\int_a^{b'} dx \int_\alpha^\beta f dy = \int_\alpha^\beta dy \int_a^{b'} f dx, \qquad b - \delta < b' < b. \tag{1}$$

$$= \int_\alpha^\beta \int_a^b - \int_\alpha^\beta \int_{b'}^b, \qquad \text{since } K \text{ is convergent.}$$

$$= K - \int_\alpha^\beta \int_{b'}^b. \tag{2}$$

But, however small $\epsilon>0$ is taken, we may take $\delta>0$ so small that

$$\left|\int_a^\beta\int_{b'}^b\right|<\epsilon, \quad \text{by 620, 4.} \tag{3}$$

From 1), 2), 3) we have

$$\left|\int_a^{b'}\int_a^\beta - K\right|<\epsilon, \qquad b-\delta<b'<b.$$

But then

$$L=\int_a^b\int_a^\beta=\lim_{b'=b}\int_a^{b'}\int_a^\beta=K.$$

2. *Let $f(xy)$ be simply regular in $R=(ab\alpha\beta)$, except on $x=a_1 \cdots$, $y=\alpha_1 \cdots$*

1°. *Let the singular integrals relative to the lines $x=a_1 \cdots$ be normal in $\mathfrak{B}$.*

2°. *Let the singular integrals relative to the lines $y=\alpha_1 \cdots$ be uniformly evanescent in any interval of $\mathfrak{A}$ not embracing the points $a_1 \cdots$*

Then the integrals

$$\int_\alpha^\beta dy\int_a^b f dx, \qquad \int_a^b dx\int_\alpha^\beta f dy$$

are convergent and equal.

For condition 2° of 1 is satisfied by 622, 2 if we replace x by y in that theorem. Condition 3° is fulfilled by 626, 2.

628. *Example.* As an application of 627, 2 let us consider

$$\int_0^\beta dy\int_0^b \frac{dx}{\sqrt{xy}}, \qquad b, \beta>0.$$

The singular lines are $x=0$, $y=0$.

The singular integral relative to $x=0$,

$$S=\int_0^a \frac{dx}{\sqrt{xy}}$$

is normal in $\mathfrak{B}=(0, \beta)$. For,

$$S=\frac{1}{\sqrt{y}}\int_0^a \frac{dx}{\sqrt{x}}=\epsilon'\sigma(y),$$

setting

$$\epsilon'=\int_0^a \frac{dx}{\sqrt{x}}, \quad \sigma(y)=\frac{1}{\sqrt{y}}.$$

Here, for any $\epsilon > 0$, there exists a $\delta > 0$ such that $\epsilon' < \epsilon$, for any $0 < a < \delta$. On the other hand, σ is integrable in $\mathfrak{B}$.

The singular integral

$$T = \int_0^\alpha \frac{dy}{\sqrt{xy}}$$

relative to $y = 0$ is uniformly evanescent in any (a, b), $a > 0$. For,

$$T \leq \frac{1}{\sqrt{a}} \int_0^\alpha \frac{dy}{\sqrt{y}}.$$

But for any $\epsilon > 0$ there exists an $\alpha_0 > 0$, such that

$$\int_0^\alpha \frac{dy}{\sqrt{y}} < \epsilon\sqrt{a}, \qquad 0 < \alpha < \alpha_0.$$

Hence

$$T < \epsilon \qquad \text{for any } \alpha < \alpha_0.$$

Thus the conditions of 627, 2 being fulfilled, both integrals

$$\int_0^\beta dy \int_0^b \frac{dx}{\sqrt{xy}}, \qquad \int_0^b dx \int_0^\beta \frac{dy}{\sqrt{xy}} \tag{1}$$

are convergent and equal.

This result is easily verified by actually evaluating these integrals. For,

$$\int_0^b \frac{dx}{\sqrt{xy}} = \frac{1}{\sqrt{y}} \int_0^b \frac{dx}{\sqrt{x}} = \frac{2\sqrt{b}}{\sqrt{y}}.$$

Hence

$$\int_0^\beta \int_0^b = 2\sqrt{b} \int_0^\beta \frac{dy}{\sqrt{y}} = 4\sqrt{b\beta}\text{; etc.}$$

Differentiation

629. *Let $f(xy)$, $f'_y(xy)$ be regular in $R = (ab\alpha\beta)$, except on the lines $x = a_1, \cdots x = a_r$.*

1°. *Let f'_y be a semi-uniformly continuous function of y in $\mathfrak{B} = (\alpha\beta)$.*

2°. *In the elementary rectangles $(a_\iota - \tau,\ a_\iota + \tau,\ \alpha,\ \beta)$, let*

$$|f'_y(xy)| \leq \phi_\iota(x), \qquad \iota = 1, 2, \cdots r,$$

the ϕ_ι's being integrable in $(a_\iota - \delta,\ a_\iota + \delta)$.

Then

$$\frac{dJ}{dy} = \frac{d}{dy}\int_a^b f(xy)\,dx = \int_a^b f'_y(xy)\,dx, \qquad \text{in } \mathfrak{B}. \tag{1}$$

For simplicity, let $x=b$ be the only singular line, and let f'_y be uniformly continuous in $\mathfrak{A}=(a, b)$, except at b. Then

$$\frac{\Delta J}{\Delta y}=\int_a^b \frac{f(x, y+h)-f(x, y)}{h}\, dx, \qquad |h|<\delta$$

$$=\int_a^b f'_y(x, \eta)dx, \qquad \text{by Law of Mean}$$

$$=\int_a^b f'_y(xy)dx+\int_a^b \{f'_y(x\eta)-f'_y(xy)\}dx$$

$$=\int_a^b f'_y(xy)dx+D. \tag{2}$$

But
$$D=\int_a^b \{f'_y(x\eta)-f'_y(xy)\}dx=\int_a^{b'}+\int_{b'}^b=D_1+D_2.$$

Also, by 2°,
$$|f'_y(x\eta)-f'_y(xy)|\leq 2\,\phi(x).$$

Hence
$$|D_2|\leq 2\int_{b'}^b \phi dx<\frac{\epsilon}{2},$$

provided b' is taken sufficiently near b.

On the other hand, f'_y being uniformly continuous in $\mathfrak{A}'=(a, b')$, we can take δ so small that

$$|f'_y(x, \eta)-f'_y(x, y)|<\frac{\epsilon}{2(b-a)}, \qquad \text{in } \mathfrak{A}'.$$

Then
$$|D_1|<\epsilon/2.$$

Hence $|D|<\epsilon$, for any $|h|<\delta$; and 1) follows at once from 2).

630. *Example.* As an application of 629, let us consider the integral

$$\int_0^1 x^{y-1}\log^n x\, dx, \qquad n \gtreqless 0, \textit{ integral},$$

which was taken up in 614. The integrand

$$f(xy)=x^{y-1}\log^n x$$

is not defined for $x=0$. Let us give it the value 0, when $x=0$. Then

$$f'_y(x, y)=x^{y-1}\log^{n+1}x \qquad \text{for } x>0$$

$$=0 \qquad \text{for } x=0.$$

Then f and f'_y are simply regular in $R=(0,\ 1,\ \alpha,\ \beta)$, $\alpha>0$ except on the line $x=0$. Moreover, f'_y is a uniformly continuous function of y in $\mathfrak{B}=(\alpha,\ \beta)$, except possibly on the line $x=0$. It is therefore a regularly continuous function of y in $\mathfrak{B}$. Thus condition 1° of 629 is fulfilled. Condition 2° is also satisfied, as 614, 2) shows. Hence if we set

$$J=\int_0^1 x^{y-1}\,dx,$$

we get, by 629,

$$\frac{dJ}{dy}=\int_0^1 x^{y-1}\log x\,dx\,;$$

or differentiating n times,

$$\frac{d^nJ}{dy^n}=\int_0^1 x^{y-1}\log^n x\,dx. \qquad (1$$

But, by 575, Ex. 3,

$$J=\frac{1}{y}.$$

Hence

$$\frac{d^nJ}{dy^n}=(-1)^n\frac{n\,!}{y^{n+1}}. \qquad (2$$

Comparing 1), 2), we get

$$\int_0^1 x^{y-1}\log^n x\,dx=(-1)^n\frac{n\,!}{y^{n+1}}, \qquad y>0. \qquad (3$$

631. 1. By using double integrals, we can obtain more general conditions than those given in 629, for differentiating under the integral sign in

$$J=\int_a^b f(xy)dx.$$

For example, the following.

Let $f(x,\ y)$, $f'_y(x,\ y)$ *be in general regular with respect to* x *in* $R=(ab\alpha\beta)$.

1°. *Let* $f'_y(xy)$ *be a semi-uniformly continuous function of* y *in* $\mathfrak{B}=(\alpha,\ \beta)$.

2°. *Let*

$$g(y)=\int_a^b f'_y(xy)dx$$

be uniformly convergent in $\mathfrak{B}$.

3°. *Let*

$$\int_y^{y+h}dy\int_a^b f'_y(x,\ y)dx, \qquad |h|<\delta$$

admit inversion.

Then

$$\frac{dJ}{dy}=\int_a^b f'_y(xy)dx, \qquad \text{in } \mathfrak{B}. \qquad (1$$

For,
$$\frac{\Delta J}{\Delta y}=\frac{1}{h}\left\{\int_a^b [f(x, y+h)-f(x, y)]dx\right\}$$

$$=\frac{1}{h}\int_a^b dx\int_y^{y+h} f'_y(x, y)dy, \qquad \text{by 605, 610, 6}$$

$$=\frac{1}{h}\int_y^{y+h} dy\int_a^b f'_y dx, \qquad \text{by } 3^\circ$$

$$=\frac{1}{h}\int_y^{y+h} g(y)dy. \tag{2}$$

But by 1°, 2°, and 618, 1, $g(y)$ is continuous in $\mathfrak{B}$. Thus on passing to the limit $h=0$, in 2), we get 1), using 537, 2.

2. As a corollary of 1 we have:

Let $f(xy)$, $f'_y(xy)$ *be regular in* $R=(ab\alpha\beta)$, *except on the lines* $x=a_1, \cdots x=a_r$.

Let $f'_y(xy)$ *be continuous in* R, *except on these lines.*

Let
$$\int_a^b f'_y(x, y)dx$$
be uniformly convergent in $\mathfrak{B}=(\alpha, \beta)$.

Then
$$\frac{d}{dy}\int_a^b f(xy)dx=\int_a^b f'_y(xy)dx, \qquad \textit{in } \mathfrak{B}.$$

CHAPTER XV

IMPROPER INTEGRALS. INTERVAL OF INTEGRATION INFINITE

Definitions

632. 1. If $f(x)$ has no points of infinite discontinuity in $\mathfrak{A} = (a, \infty)$, and is integrable in any partial interval (a, b), of $\mathfrak{A}$, we shall say that *$f(x)$ is regular in* $\mathfrak{A}$. If on the contrary, $f(x)$ has a finite number of points of infinite discontinuity $c_1, c_2, \cdots$ in $\mathfrak{A}$, but is integrable in any partial interval (a, b), we shall say *$f(x)$ is in general regular in* $\mathfrak{A}$. The points $c_1, c_2, \cdots$ are *singular points.*

Let $f(x)$ be in general regular in $\mathfrak{A}$. Let us consider

$$\lim_{x=\infty}\int_a^x f(x)dx, \tag{1}$$

which we denote more shortly by

$$\int_a^\infty f(x)dx, \tag{2}$$

and which is called the *integral of $f(x)$ from a to* $+\infty$, or the *integral of $f(x)$ in* $\mathfrak{A}$.

If the limit 1) is finite, we say the integral 2) is *convergent* or that *$f(x)$ is integrable in* $\mathfrak{A}$. If the limit 1) is infinite, the integral 2) is *divergent.* If the limit 1) does not exist, *i.e.* if it is neither finite nor definitely infinite, the integral 2) *does not exist.*

If $|f(x)|$ is integrable in $\mathfrak{A}$, $f(x)$ is *absolutely integrable* in $\mathfrak{A}$, and the integral 2) is *absolutely convergent.*

2. We make a remark here which will often enable us to simplify our demonstrations, without loss of generality.

If $f(x)$ is *in general* regular in $\mathfrak{A} = (a, \infty)$, we can take b so large that $f(x)$ is *regular* in (b, ∞). But the integral

$$\int_a^b f(x)dx$$

has been treated in Chapter XIV. We have thus only to consider

$$\int_b^\infty f(x)dx,$$

in which the integrand is regular. We may therefore often assume in our demonstration, without loss of generality, that $f(x)$ is regular in $\mathfrak{A}$.

Ex. 1.

$$\int_1^\infty \frac{dx}{x^2} = 1; \qquad \text{it is convergent.}$$

For,

$$\lim_{x=\infty} \int_1^x \frac{dx}{x^2} = \lim \left(1 - \frac{1}{x}\right) = 1.$$

Ex. 2.

$$\int_1^\infty \frac{dx}{x} = +\infty; \qquad \text{it is divergent.}$$

For,

$$\lim_{x=\infty} \int_1^x \frac{dx}{x} = \lim \log x = +\infty.$$

Ex. 3.

$$\int_0^\infty \cos x\, dx \qquad \text{does not exist.}$$

For,

$$\lim_{x=\infty} \int_0^x \cos x\, dx = \lim \sin x$$

does not exist.

633. 1. *General criterion for convergence.*

Let $f(x)$ *be regular in* $\mathfrak{A} = (a, \infty)$. *For*

$$\int_a^{+\infty} f dx$$

to be convergent, it is necessary and sufficient that, for each $\epsilon > 0$, *there exists a* $G > 0$, *such that*

$$\left| \int_\alpha^\beta f dx \right| < \epsilon \qquad (2$$

for any pair of numbers, $\alpha, \beta \gtreqless G$

This is a direct consequence of 284.

2. The integral 2) is perfectly analogous to the singular integrals considered in Chapter XIV. It is convenient to call it *the singular integral relative to the point* $x=\infty$; and also to call this point *a singular point.*

Instead of 2) it is convenient at times to call

$$\int_b^\infty f(x)dx, \qquad b>G, \tag{3}$$

the singular integral. The integrals 2) and 3) are obviously equivalent.

3. *Let* $f(x)$ *be regular in* $\mathfrak{A}=(\alpha, \infty)$. *If* $f(x)$ *is absolutely integrable in* $\mathfrak{A}$, *it is also integrable in* $\mathfrak{A}$.

For by hypothesis

$$\int_\alpha^\beta |f(x)|dx<\epsilon, \qquad G<\alpha<\beta.$$

But in any given (α, β), $f(x)$ is integrable by hypothesis and

$$\left|\int_\alpha^\beta f dx\right| \leq \int_\alpha^\beta |f|dx, \qquad \text{by 528, 1.}$$

4. The reader should note that an integral may be convergent and yet *not* absolutely convergent. Thus

$$\int_0^\infty \frac{\sin x}{x^\mu}dx$$

converges for $0<\mu\leq 1$, while

$$\int_0^\infty \frac{|\sin x|}{x^\mu}dx$$

is divergent for these values of μ. Cf. 646, Ex. 2.

634. 1. The symbol

$$\int_{-\infty}^a f(x)dx \tag{1}$$

is defined as

$$\lim_{\alpha=-\infty} \int_\alpha^a f(x)dx,$$

and is called the integral of $f(x)$ from $-\infty$ to a.

The symbol

$$\int_{-\infty}^{\infty} f(x)dx \tag{2}$$

is defined as

$$\lim_{\substack{\alpha=-\infty \\ \beta=+\infty}} \int_{\alpha}^{\beta} f(x)dx.$$

The terms introduced in 632 have a similar meaning here. The integral 1) is not essentially different from

$$\int_{a}^{\infty} f(x)dx,$$

and requires therefore no comment.

2. In regard to 2), we have the theorem:

Let $f(x)$ *be in general regular in* $(-\infty, \infty)$. *In order that*

$$J = \int_{-\infty}^{\infty} f dx$$

be convergent, it is necessary and sufficient that

$$J_1 = \int_{-\infty}^{a} f dx, \quad J_2 = \int_{a}^{\infty} f dx$$

be convergent for any a. *When* J *is convergent,*

$$J = J_1 + J_2.$$

If J is convergent, we have

$$\epsilon > 0, \quad G > 0, \quad \left| J - \int_{\alpha}^{\beta} \right| < \frac{\epsilon}{2}, \quad \alpha < -G, \quad \beta > G.$$

$$\left| J - \int_{\alpha'}^{\beta} \right| < \frac{\epsilon}{2}, \quad \alpha' < -G.$$

Subtracting, we get

$$\left| \int_{\alpha}^{\beta} - \int_{\alpha'}^{\beta} \right| < \epsilon,$$

or

$$\left| \int_{\alpha}^{\alpha'} f dx \right| < \epsilon.$$

Hence, by 633,

$$\int_{-\infty}^{a} f dx$$

is convergent. Similarly,

$$\int_{a}^{\infty} f dx$$

is convergent. Therefore when J is convergent, J_1, J_2 are convergent.

Conversely, if J_1, J_2 are convergent, J is convergent.

For, let a be fixed and $\alpha < a < \beta$. Then

$$\int_{\alpha}^{\beta} = \int_{\alpha}^{a} + \int_{a}^{\beta}, \quad \text{by 593.}$$

For a sufficiently large G, and $\epsilon > 0$ arbitrarily small,

$$\int_{\alpha}^{a} = J_1 + \epsilon', \qquad \alpha < -G, \ |\epsilon'| < \frac{\epsilon}{2}.$$

$$\int_{a}^{\beta} = J_2 + \epsilon'', \qquad \beta > G, \ |\epsilon''| < \frac{\epsilon}{2}.$$

Hence

$$\left| \int_{\alpha}^{\beta} - (J_1 + J_2) \right| < \epsilon,$$

for all $\alpha < -G$, and $\beta > G$. Therefore

$$\lim \int_{\alpha}^{\beta} = J_1 + J_2.$$

Hence, when J_1, J_2 are convergent, J is, and

$$J = J_1 + J_2.$$

3. As a result of the foregoing, we see that the integrals

$$\int_{-\infty}^{a} f dx, \quad \int_{-\infty}^{\infty} f dx \tag{3}$$

do not differ essentially from

$$\int_{a}^{\infty} f dx.$$

For convenience, we shall therefore study only this last; the results we obtain are then readily extended to the integrals 3).

Tests for Convergence

635. **1.** *The* μ *tests for convergence. Let* $f(x)$ *be regular in* $\mathfrak{A}=(a, \infty)$. *If there exists a* $\mu>1$, *such that*

$$x^{\mu}|f(x)| \leq M, \quad M>0, \qquad in\, V(\infty);$$

$f(x)$ *is absolutely integrable in* $\mathfrak{A}$.

For let $G<\alpha<\beta$ lie in V. Then, by 526, 2,

$$\left|\int_{\alpha}^{\beta} f dx\right| \leq \int_{\alpha}^{\beta}|f| dx \leq \int_{\alpha}^{\beta} \frac{M}{x^{\mu}} dx=\frac{M}{\mu-1}\left\{\frac{1}{\alpha^{\mu-1}}-\frac{1}{\beta^{\mu-1}}\right\},$$

$<\epsilon$, if G is taken sufficiently large.

2. *Let* $f(x)$ *be regular in* $\mathfrak{A}=(a, \infty)$. *If for some* $\mu>1$

$$\lim_{x=\infty} x^{\mu}|f(x)|$$

is finite, $f(x)$ *is absolutely integrable in* $\mathfrak{A}$.

3. As corollary of 2 we have:

In $\mathfrak{A}=(a, \infty)$, $a>0$, let

$$f(x)=\frac{g(x)}{x^{\lambda}} \text{ or } \frac{x^{a} g(x)}{e^{\nu x}}, \qquad \lambda>1 ;\ \nu>0,$$

where g *is limited in* $\mathfrak{A}$ *and integrable in any* (a, b). *Then* $f(x)$ is *absolutely integrable in* $\mathfrak{A}$.

636. *Test for divergence. Let* $f(x)$ *be regular in* $\mathfrak{A}=(a, \infty)$. *In* $V(\infty)$ *let* f *have one sign* σ, and

$$\sigma x f(x)>M, \qquad M>0.$$

Then

$$J=\int_{a}^{\infty} f dx=\sigma \cdot \infty.$$

For, let $a<\alpha<x$; while α, x lie in V. Then

$$\int_{a}^{x}=\int_{a}^{\alpha}+\int_{\alpha}^{x}.$$

Now, by 526, 2,

$$\int_{\alpha}^{x} \sigma f dx \geq M \int_{\alpha}^{x} \frac{dx}{x}=M \log \frac{x}{\alpha} \doteq+\infty,$$

when $x=+\infty$.

637. *Logarithmic test for convergence. Let $f(x)$ be regular in $\mathfrak{A} = (a, \infty)$. If there exists a $\mu > 1$, and an s, such that*

$$xl_1xl_2x \cdots l_{s-1}xl_s{}^{\mu}x\,|f(x)| < M, \qquad \textit{in } V(\infty),$$

$f(x)$ is absolutely integrable in $\mathfrak{A}$.

We have, by 389, 2), for $x > 0$ sufficiently large,

$$D_x l_s^{1-\mu}x = \frac{1-\mu}{xl_1xl_2x \cdots l_{s-1}xl_s{}^{\mu}x}.$$

Hence, if $0 < G < \alpha < \beta$,

$$\int_\alpha^\beta \frac{dx}{xl_1x \cdots l_s{}^{\mu}x} = \frac{1}{\mu - 1}\left\{\frac{1}{l_s{}^{\mu-1}\alpha} - \frac{1}{l_s{}^{\mu-1}\beta}\right\} \doteq 0,$$

when $G \doteq +\infty$.

Now

$$\left|\int_\alpha^\beta f dx\right| \leq \int_\alpha^\beta |f| dx \leq M\int_\alpha^\beta \frac{dx}{xl_1x \cdots l_s{}^{\mu}x}$$

$< \epsilon$, for G sufficiently large.

638. *The logarithmic test for divergence. Let $f(x)$ be regular in $\mathfrak{A} = (a, \infty)$. In $V(\infty)$, let $f(x)$ have one sign σ, and*

$$xl_1xl_2x \cdots l_sx \cdot \sigma f(x) > M > 0.$$

Then

$$\int_a^\infty f dx = \sigma \cdot \infty.$$

For,

$$\int_a^x = \int_a^\alpha + \int_\alpha^x.$$

But

$$\int_\alpha^x \sigma f dx > M\int_\alpha^x \frac{dx}{xl_1x \cdots l_sx} = Ml_{s+1}\frac{x}{\alpha},$$

since by 389, 1), for sufficiently large $x > 0$,

$$D_x l_{s+1}x = \frac{1}{xl_1x \cdots l_sx}, \qquad l_0 = 1.$$

As

$$\lim_{x=+\infty} l_{s+1}\frac{x}{\alpha} = +\infty,$$

our theorem is established.

639. Ex. 1. A quarter period of Jacobi's function $sn(u, \kappa)$ is

$$K = \int_{1/\kappa}^{\infty} \frac{dx}{\sqrt{(1-x^2)(1-\kappa^2 x^2)}}. \qquad 0<\kappa<1.$$

The point $x = 1/\kappa$ is the only finite singular point. Applying the μ test of 579 at this point, we see that we can take $\mu = \frac{1}{2}$. The singular integral at this point is therefore evanescent. Consider now the point $x = \infty$. We have

$$\frac{1}{\sqrt{(1-x^2)(1-\kappa^2x^2)}} = \frac{1}{x^2\sqrt{\left(1-\frac{1}{x^2}\right)\left(\kappa^2-\frac{1}{x^2}\right)}}.$$

This shows that the μ test of 635 is satisfied for any $\mu \leqq 2$. Hence the singular integral relative to this point is evanescent. Hence K is convergent.

640. Ex. 2. A half period of Weierstrass's function $\wp(u, g_2, g_3)$ is

$$\omega = \int_{e_1}^{\infty} \frac{dx}{\sqrt{4x^3 - g_2x - g_3}},$$

where e_1 is the largest real root of

$$4x^3 - g_2x - g_3.$$

The point e_1 is the only finite singular point. Applying the μ test of 579 at this point, we see that we can take $\mu = \frac{1}{2}$. Hence the singular integral relative to this point is evanescent. Consider the point $x = \infty$. We have

$$\frac{1}{\sqrt{4x^3 - g_2x - g_3}} = \frac{1}{x^{\frac{3}{2}}\sqrt{4 - \frac{g_2}{x^2} - \frac{g_3}{x^3}}}.$$

This shows that the μ test of 635 is satisfied for any $\mu \geqq \frac{3}{2}$. Hence ω is convergent.

641. Ex. 3. *The Beta function,*

$$\mathrm{B}(u, v) = \int_0^{\infty} \frac{x^{u-1}dx}{(1+x)^{u+v}}. \tag{1}$$

The point $x = 0$ is a singular point if $u < 1$. Applying the tests of 579, 580 at this point, we see that

$$\int_0^a \frac{x^{u-1}dx}{(1+x)^{u+v}}, \qquad a>0,$$

is convergent when $u > 0$; and divergent, when $u \leqq 0$. Consider the point $x = \infty$. We have

$$f(x) = \frac{x^{u-1}}{(1+x)^{u+v}} = \frac{1}{x^{v+1}} \cdot \frac{1}{\left(1+\frac{1}{x}\right)^{u+v}}.$$

Applying now the tests of 635, 636, we see that

$$\int_a^\infty f(x)dx, \qquad u>0.$$

converges for $v>0$, and diverges for $v \overset{=}{<} 0$.

Thus the integral 1) has a finite value for every u, $v>0$. The function so defined is called the *Eulerian integral of the first kind* or the *Beta function.*

642. Ex. 4. *The Gamma function,*

$$\Gamma(u)=\int_0^\infty e^{-x}x^{u-1}dx. \tag{1}$$

The point $x=0$ is a singular point if $u<1$.

Applying the tests 579, 580 at this point, we see that

$$\int_0^a, \qquad a>0.$$

is convergent when $u>0$, and divergent when $u \overset{=}{<} 0$.

On the other hand, applying the μ test of 635, 3, we see that

$$\int_a^\infty$$

is convergent for any u. The integral 1) therefore defines a function of u for all $u>0$. It is called the *Eulerian Integral of the second kind* or the *Gamma function.*

643. 1. *Let $f(x)$ be regular and integrable in any partial interval (a, b), of $\mathfrak{A}=(a, \infty)$.*

Let
$$\int_a^x f(x)dx$$
be limited in $\mathfrak{A}$.

In $V(\infty)$, let $g(x)$ be monotone, and $g(\infty)=0$.

Then $f(x)g(x)$ is integrable in $\mathfrak{A}$.

We apply the criterion of 633. Let $G<\alpha<\beta$ lie in $V(\infty)$. Then by the Second Theorem of the Mean, 545,

$$\int_\alpha^\beta fgdx = g(\alpha+0)\int_\alpha^\xi fdx + g(\beta-0)\int_\xi^\beta fdx, \qquad \alpha \leq \xi \leq \beta.$$

We can take G so large that

$$g(\alpha+0), \qquad g(\beta-0)$$

are numerically as small as we please. As the integrals on the right are numerically less than some fixed number, we have

$$\left|\int_\alpha^\beta fgdx\right| < \epsilon$$

for any pair of numbers α, $\beta>G$. Hence fg is integrable in $\mathfrak{A}$.

2 *Let* $f(x)$ *be regular and integrable in* $\mathfrak{A} = (a, \infty)$. *Let* $g(x)$ *be limited and monotone in* $\mathfrak{A}$.

Then fg *is integrable in* $\mathfrak{A}$.

For, by 545,

$$\int_a^\beta fg dx = g(\alpha + 0)\int_a^\xi f dx + g(\beta - 0)\int_\xi^\beta f dx. \tag{1}$$

Let

$$|g(x)| < M.$$

Since f is integrable in $\mathfrak{A}$, we can take γ so large that

$$\left|\int_a^\xi f dx\right|, \qquad \left|\int_\xi^\beta f dx\right| < \frac{\epsilon}{2M}; \qquad \alpha, \beta > \gamma.$$

Then the right side of 1) is numerically $< \epsilon$. Hence fg is integrable.

3. *Let* $f(x)$ *be regular and absolutely integrable in* $\mathfrak{A} = (a, \infty)$. *Let* $g(x)$ *be limited and integrable in* $\mathfrak{A}$. *Then* $f(x)g(x)$ *is absolutely integrable in* $\mathfrak{A}$.

We have only to show by 633 that

$$\epsilon > 0, \qquad G > 0, \qquad \int_a^\beta |fg| dx < \epsilon, \tag{1}$$

for any pair of numbers $\alpha, \beta > G$.

Now $g(x)$ being limited, we have

$$|g(x)| \leq M, \qquad \text{in } \mathfrak{A}.$$

Hence

$$\int_a^\beta |fg| dx \leq M\int_a^\beta |f| dx. \tag{2}$$

But $f(x)$ being absolutely integrable in $\mathfrak{A}$, we can take G so large that

$$\int_a^\beta |f| dx < \frac{\epsilon}{M}, \qquad \alpha, \beta > G.$$

This in 2) gives 1).

644. *Let* $f(x)$ *be in general regular in* $\mathfrak{A} = (a, \infty)$, *but not integrable in* $\mathfrak{A}$. *Let*

$$\int_a^x f(x) dx \tag{1}$$

be limited in $\mathfrak{A}$. *Let* $g(x)$ *be monotone in* $\mathfrak{A}$ *and* $g(\infty) = G \neq 0$. *Then*

$$\int_a^\infty f(x)g(x)dx \tag{2}$$

is not convergent.

For, if 2) were convergent, we would have

$$\epsilon > 0, \qquad b, \qquad \left|\int_\alpha^\beta fgdx\right| < \epsilon, \qquad \alpha,\ \beta > b.$$

But this is impossible. For

$$\int_\alpha^\beta fgdx = g(\alpha + 0)\int_\alpha^\xi fdx + g(\beta - 0)\int_\xi^\beta fdx. \tag{3}$$

Let the integral 1) be numerically $< M$.

We can take b so great that

$$|g(x) - G| < \sigma, \qquad x > b,$$

where σ is small at pleasure.

We can therefore write 3)

$$\int_\alpha^\beta fgdx = (G + \sigma')\int_\alpha^\xi fdx + (G + \sigma'')\int_\xi^\beta fdx, \qquad |\sigma'|,\ |\sigma''| < \sigma.$$

Hence

$$\left|\int_\alpha^\beta fdx\right| \leq \frac{\epsilon + 4\sigma M}{|G|} < \epsilon', \qquad \epsilon' \text{ small at pleasure,}$$

which states that $f(x)$ is integrable in $\mathfrak{A}$.

645. Ex. 1. For what values of μ does

$$J = \int_0^\infty \frac{\cos x}{x^\mu} dx$$

converge?

Set

$$J_1 = \int_0^a \frac{\cos x}{x^\mu} dx, \qquad J_2 = \int_a^\infty \frac{\cos x}{x^\mu} dx; \qquad a > 0.$$

The integral J_2 is convergent by 643, 1, provided $\mu > 0$. For $\mu \leqq 0$, it obviously does not converge. The integral J_1 is convergent, as we saw, 586, only when $\mu < 1$.

Thus J is convergent when and only when

$$0 < \mu < 1.$$

646. Ex. 2.

$$J = \int_0^\infty \frac{\sin x}{x^\mu}\, dx.$$

Set

$$J_1 = \int_0^a \frac{\sin x}{x^\mu}\, dx, \qquad J_2 = \int_a^\infty \frac{\sin x}{x^\mu}\, dx\,; \qquad a > 0.$$

In 587, we saw J_1 is convergent only when $\mu < 2$.

By 643, 1, J_2 is convergent when $\mu > 0$. When $\mu \leqq 0$ it obviously does not converge.

Hence J is convergent when, and only when,

$$0 < \mu < 2.$$

That J does not converge absolutely for $0 < \mu \leq 1$, is shown as follows. We have

$$K_n = \int_0^{n\pi} \frac{|\sin x|\, dx}{x^\mu} = \int_0^\pi + \int_\pi^{2\pi} + \cdots + \int_{(n-1)\pi}^{n\pi}$$

$$= J_1 + J_2 + \cdots + J_n.$$

Let

$$x = y + (m-1)\pi.$$

Then

$$J_m = \int_0^\pi \frac{\sin y\, dy}{\{y + (m-1)\pi\}^\mu} > \frac{1}{(m\pi)^\mu} \int_0^\pi \sin y\, dy = \frac{2}{(m\pi)^\mu}.$$

Hence

$$K_n > \frac{2}{\pi^\mu} \left\{ \frac{1}{1^\mu} + \frac{1}{2^\mu} + \cdots + \frac{1}{n^\mu} \right\} \doteq \infty,$$

when $n \doteq \infty$, as the reader probably knows, or as will be shown later.

Properties of Integrals

647. In Chapter XIV we established the properties of the improper integrals,

$$\int_a^b f dx,$$

by a passage to the limit. We propose now to develop the properties of improper integrals, the interval of integration being infinite, by a similar method. In many cases the reasoning is so similar to that employed to prove the corresponding theorems in Chapter XIV, that we shall not repeat it, referring the reader to the demonstrations given in that chapter.

648. 1. *Let $f(x)$ be integrable in (a, ∞). Then*

$$\int_a^\infty f dx = -\int_\infty^a f dx.$$

2. *Let $f(x)$ be integrable in (a, ∞). Then*

$$\int_a^\infty f dx = \int_a^b f dx + \int_b^\infty f dx, \qquad a < b.$$

3. *Let $f_1(x) \cdots f_m(x)$ be integrable in (a, ∞). Then*

$$\int_a^\infty (c_1 f_1 + \cdots + c_m f_m) dx = c_1 \int_a^\infty f_1 dx + \cdots c_m \int_a^\infty f_m dx.$$

649. 1. *Let $f(x)$, $g(x)$ be integrable in (a, ∞). Except possibly at the singular points let $f(x) \gtreqless g(x)$. Then*

$$\int_a^\infty f dx \geq \int_a^\infty g dx.$$

2. *Let $f(x)$, $g(x)$ be integrable in (a, ∞). Except possibly at the singular points, let $f(x) \gtreqless g(x)$. At a point c of continuity of these functions, let $f(c) > g(c)$. Then*

$$\int_a^\infty f dx > \int_a^\infty g dx.$$

3. *Let $f(x) \gtreqless 0$ be integrable in (a, ∞). At a point c of continuity of f, let $f(c) > 0$. Then*

$$\int_a^\infty f(x) dx > 0.$$

4. *Let $f(x)$ be absolutely integrable in (a, ∞). Then*

$$\left| \int_a^\infty f dx \right| \leq \int_a^\infty |f| dx.$$

5. *Let*

$$J = \int_a^\infty f dx$$

be convergent. We may change the value of $f(x)$ over a limited discrete aggregate, without altering the value of J, provided the new values of f are limited.

650. *Let $f(x)$ be integrable in $\mathfrak{A} = (a, \infty)$. Then*

$$J(x) = \int_x^\infty f dx \qquad a \leq x.$$

is a continuous limited function of x in $\mathfrak{A}$.

For, by 648, 2,

$$\int_x^\infty = \int_x^c + \int_c^\infty, \qquad c > x.$$

But

$$\int_x^c f dx$$

is a continuous function of x in (a, c), by 603. As

$$\int_c^\infty f dx$$

is a constant, $J(x)$ is continuous in $\mathfrak{A}$.

J is limited in $\mathfrak{A}$. In fact, for each $\epsilon > 0$, there exists a c such that

$$\left| \int_c^\infty f dx \right| < \epsilon.$$

But

$$\int_x^c f dx,$$

being continuous in the limited interval (a, c), is limited. Hence J is limited in $\mathfrak{A}$.

651. *Let $f(x)$ be integrable in $\mathfrak{A} = (a, \infty)$. Then*

$$\frac{d}{dx} \int_x^\infty f dx = -f(x)$$

for any point x of $\mathfrak{A}$, at which $f(x)$ is continuous.

For, if $c > x$,

$$J(x) = \int_x^\infty = \int_x^c + \int_c^\infty = K(x) + C,$$

C being a constant. By 604, 1,

$$\frac{dK}{dx} = -f(x).$$

Hence

$$\frac{dJ}{dx} = \frac{d(K + C)}{dx} = \frac{dK}{dx} = -f(x).$$

652. *In $\mathfrak{A} = (a, \infty)$, let $f(x)$ be continuous excepting possibly at certain points $c_1 \cdots c_s$, where it may be unlimited. Let it be integrable in any (a, b).*

Let $F(x)$ be one-valued and continuous in $\mathfrak{A}$; having $f(x)$ as derivative except at the points c.

Then

$$\int_a^\infty f dx = F(+\infty) - F(a), \tag{1}$$

where $F(+\infty)$ is finite or infinite.

For, by 605,

$$\int_a^b f dx = F(b) - F(a),$$

however large b is. Passing to the limit, we get 1).

Theorems of the Mean

653. *First Theorem of the Mean. In $\mathfrak{A} = (a, \infty)$ let $g(x)$ be integrable and limited.*

Let $f(x)$ be integrable, and non-negative in $\mathfrak{A}$. Then

$$\int_a^\infty fg dx = \mathfrak{M} \int_a^\infty f dx, \tag{1}$$

where $\mathfrak{M}$ is a mean value of g in $\mathfrak{A}$.

For, by 602,

$$m \int_a^b f dx \leq \int_a^b fg dx \overset{\equiv}{<} M \int_a^b f dx, \qquad a < b. \tag{2}$$

where

$$m \leq g(x) \leq M.$$

Let $b \doteq +\infty$; since all the integrals in 2) are convergent, by 643, 3, we get in the limit,

$$m \int_a^\infty f dx \leq \int_a^\infty fg dx \leq M \int_a^\infty f dx,$$

which gives 1).

654. *Second Theorem of the Mean. In $\mathfrak{A} = (a, \infty)$, let $f(x)$ be integrable and $g(x)$ limited and monotone. Then*

$$J = \int_a^\infty f(x) g(x) dx = g(a+0) \int_a^\eta f(x) dx + g(\infty) \int_\eta^\infty f(x) dx, \tag{1}$$

$$a \leq \eta \leq \infty.$$

If $g(a+0)=g(+\infty)$ the theorem is obviously true. We may therefore assume that these limits are different. Next we observe that the integral J is convergent, by 599, 2 and 643, 2.

To fix the ideas, let $g(x)$ be monotone increasing.

Let b be arbitrarily large. Then, by 608,

$$\int_a^b fg dx = g(a+0)\int_a^\xi f dx + g(b-0)\int_\xi^b f dx. \qquad (2$$

Let us add

$$B = g(+\infty)\int_b^\infty f dx$$

to both sides of 2); observing that

$$\lim_{b=+\infty} B = 0. \qquad (3$$

We get

$$\int_a^b fg dx + B = g(a+0)\int_a^\xi f dx + g(b-0)\int_\xi^b f dx + g(+\infty)\int_b^\infty fax$$

$$= g(a+0)\int_a^\infty f dx - g(a+0)\int_\xi^\infty f dx + g(b-0)\int_\xi^\infty f dx$$

$$- g(b-0)\int_b^\infty f dx + g(+\infty)\int_b^\infty f dx$$

$$= g(a+0)\int_a^\infty f dx + \left\{g(b-0) - g(a+0)\right\}\int_\xi^\infty f dx$$

$$+ \left\{g(+\infty) - g(b-0)\right\}\int_b^\infty f dx$$

$$= g(a+0)\int_a^\infty f dx + U + V. \qquad (4$$

Let λ, μ be the minimum and maximum of

$$\int_x^\infty f dx$$

in $\mathfrak{A}$. Then obviously,

$$\lambda \le \int_\xi^\infty f dx \le \mu,$$

$$\lambda \le \int_b^\infty f dx \le \mu.$$

Hence

$$\{g(b-0)-g(a+0)\}\lambda \leq U \leq \{g(b-0)-g(a+0)\}\mu,$$

$$\{g(+\infty)-g(b-0)\}\lambda \leq V \leq \{g(+\infty)-g(b-0)\}\mu.$$

Adding, we get

$$\{g(+\infty)-g(a+0)\}\lambda \leq U+V \leq \{g(+\infty)-g(a+0)\}\mu.$$

Hence

$$U+V=\mathfrak{M}'\{g(+\infty)-g(a+0)\}, \tag{5}$$

where

$$\lambda \leq \mathfrak{M}' \leq \mu.$$

From 4) and 5) we have

$$\int_a^b fg dx + B = g(a+0)\int_a^\infty f dx + \mathfrak{M}'\{g(+\infty)-g(a+0)\}.$$

Passing to the limit $b=\infty$, and using 3), we have

$$J=g(a+0)\int_a^\infty f dx + \mathfrak{M}\Big\{g(+\infty)-g(a+0)\Big\}; \tag{6}$$

where

$$\lim_{b=\infty} \mathfrak{M}' = \mathfrak{M},$$

and

$$\lambda \leq \mathfrak{M} \leq \mu.$$

The integral

$$\int_x^\infty f dx$$

being a continuous function of x in (a, ∞), must take on the value $\mathfrak{M}$ for some point $x=\eta$, finite or infinite, in this interval. Then

$$\mathfrak{M}=\int_\eta^\infty f dx.$$

From 6) we have now,

$$J=g(a+0)\int_a^\infty f dx + \Big\{g(+\infty)-g(a+0)\Big\}\int_\eta^\infty f dx$$

$$=g(a+0)\int_a^\eta f dx + g(+\infty)\int_\eta^\infty f dx,$$

which is 1).

Change of Variable

655. *Let* $\mathfrak{B} = (\alpha, \beta)$, $\alpha \leqq \beta$; *either* α *or* β *may be infinite.*

Let $x = \psi(u)$ *have a continuous derivative in* $\mathfrak{B}$, *which may vanish over a discrete aggregate, but has otherwise one sign.*

Let $\mathfrak{A} = (a, b)$ *be the image of* $\mathfrak{B}$, *where*

$$a = \lim_{u=\alpha} \psi(u), \quad b = \lim_{u=\beta} \psi(u);$$

and b *may be infinite.*

Let $f(x)$ *be integrable in any* (a, b'), $\qquad b' < b$.

If now, either

$$J_x = \int_a^b f(x)dx, \text{ or } J_u = \int_\alpha^\beta f[\psi(u)]\psi'(u)du$$

is convergent, the other is, and both are equal.

There are various cases. Let us take the following as illustration. Let $\mathfrak{A} = (a, \infty)$, $\mathfrak{B} = (\alpha, \infty)$. By virtue of 403, the points of $\mathfrak{A}$ and $\mathfrak{B}$ are in 1 to 1 correspondence.

Let b', β' be corresponding points in $\mathfrak{A}$, $\mathfrak{B}$. Then as $\beta' \doteq \infty$, $b' \doteq \infty$ also, and conversely.

By 606, 2,

$$\int_a^{b'} f(x)dx = \int_\alpha^{\beta'} f[\psi(u)]\psi'(u)du. \tag{1}$$

Suppose J_x is convergent. Then 1) shows, passing to the limit, that J_u is convergent and $J_x = J_u$. The supposition that J_u is convergent leads to a similar conclusion for J_x.

656. Ex. 1. Consider the convergence of

$$J = \int_0^\infty \sin x^2\, dx.$$

Since the integrand is continuous, J converges if

$$J_x = \int_1^\infty \sin x^2\, dx$$

converges. Let

$$x = \psi(u) = \sqrt{u};$$

and

$$\mathfrak{A} = (1, \infty), \quad \mathfrak{B} = (1, \infty).$$

Then

$$J_u = \int_1^\infty \frac{\sin u}{\sqrt{u}} dx,$$

which is convergent by 646.

Hence, by 655, J_x, and therefore J, is convergent.

Ex. 2. We found, by 630, 3), that

$$\int_0^1 x^{y-1} \log^n x \, dx = \frac{(-1)^n n!}{y^{n+1}}, \qquad y > 0. \tag{1}$$

Let us set

$$x = \psi(u) = e^{-u}.$$

Here

$$a = 0, \quad b = 1; \quad \alpha = +\infty, \quad \beta = 0.$$

In $\mathfrak{B}$,

$$\psi'(u) = -e^{-u}$$

is continuous and always negative.

Then, by 655,

$$J_u = -(-1)^n \int_{+\infty}^0 e^{-uy} u^n \, du \tag{2}$$

is convergent, since 1) is. Hence 1), 2) give

$$\int_0^\infty e^{-uy} u^n \, du = \frac{n!}{y^{n+1}}, \qquad y > 0. \tag{3}$$

657. 1. *Stoke's Integrals.* Let us consider the convergence of the integral

$$J = \int_0^\infty x \sin (x^3 - xy) dx, \tag{1}$$

which comes up in the theory of the Rainbow.

Let us set

$$u = x^3 - xy = x(x^2 - y) = \phi(x). \tag{2}$$

The graph of this is a curve which crosses the axes at the points $x = 0$, $x = \pm\sqrt{y}$, if $y > 0$; and at the point $x = 0$, if $y = 0$. To fix the ideas, let us suppose $y > 0$; the case when $y \overset{=}{<} 0$ may be treated in a similar manner.

Supposing, therefore, $y > 0$, the graph of 2) shows that as x ranges over $\mathfrak{A} = (\sqrt{y}, \infty)$, u ranges over $\mathfrak{B} = (0, \infty)$, the correspondence between the points of $\mathfrak{A}$ and $\mathfrak{B}$ being uniform. Thus the relation defines a one-valued inverse function $x = \psi(u)$ in $\mathfrak{B}$.

Let us write 1)

$$J = \int_0^{\sqrt{y}} + \int_{\sqrt{y}}^\infty,$$

and denote the latter integral by J_x.

The corresponding integral in u is

$$J_u = \int_0^\infty g(u) \sin u \, du,$$

setting

$$g(u) = \frac{x}{3x^2 - y}.$$

We can now apply 643, 1. For, $x \doteq +\infty$ as $u \doteq +\infty$. Hence $g(u)$ is a monotone decreasing function, for any positive y, and $g(\infty) = 0$. Thus J_u is convergent. Hence J_x is; and therefore the integral J is convergent.

2. The same considerations show *à fortiori*, that

$$K = \int_0^\infty \cos (x^3 - xy) dx \tag{3}$$

is convergent for any y.

3. In connection with these integrals, occurs another integral

$$L = \int_0^\infty x^2 \cos(x^3 - xy)dx, \tag{4}$$

which, it is important to show, is not convergent. In fact, effecting the change of variable defined by 2), in

$$L_x = \int_{\sqrt{y}}^\infty x^2 \cos(x^3 - xy)dx,$$

supposing to fix the ideas that $y > 0$, we get

$$L_u = \int_0^\infty \frac{x^2}{3x^2 - y} \cos u\, du = \int_0^\infty h(u) \cos u\, du.$$

Here $h(u)$ is a monotone function, and

$$h(\infty) = \tfrac{1}{3}.$$

Thus L_u is divergent, by 644. Hence L_x is. Therefore L is divergent.

INTEGRALS DEPENDING ON A PARAMETER

Uniform Convergence

658. 1. Let $f(x, y)$ be defined at each point of the rectangle $R = (a\infty\alpha\beta)$, β finite or infinite. Let $\mathfrak{A} = (a, \infty)$, $\mathfrak{B} = (\alpha, \beta)$.

We shall say $f(xy)$ is *regular in R* when:

1°. $f(xy)$ has no point of infinite discontinuity in R.

2°. $f(xy)$ is integrable in $\mathfrak{A}$ for each y in $\mathfrak{B}$.

At times we shall need to integrate $f(xy)$ with respect to y. In this case we shall also suppose:

3°. $f(xy)$ is integrable in $\mathfrak{B}$ for each x in $\mathfrak{A}$.

2. If $f(xy)$ is regular in R, except that it *may* have points of infinite discontinuity on certain lines $x = a_1, \cdots x = a_r$, we shall say $f(xy)$ is *regular in R except on the lines $x = a_1, \cdots$* or that it is *in general regular with respect to x.*

3. Let us suppose that the points of infinite discontinuity of $f(xy)$ do not lie all on a finite number of lines parallel to the y-axis, but that it is necessary to employ in addition a finite number of lines parallel to the x-axis. To fix the ideas, let these lines be $x = a_1, \cdots x = a_r$; $y = \alpha_1, \cdots y = \alpha_s$. If $f(xy)$ is otherwise regular in R, *i.e.* if it enjoys properties 2°, 3° of 658, we shall say

$f(xy)$ is *in general regular with respect to* x, y, or $f(xy)$ is *regular except on the lines* $x = a_1, \cdots y = \alpha_1, \cdots$

4. Let $f(xy)$ be continuous at each point of R except on certain lines $x = a_1, \cdots x = a_r$; $y = \alpha_1, \cdots y = \alpha_s$. On the lines $x = a_1, \cdots$ it *may* have points of infinite discontinuity; on the lines $y = \alpha_1, \cdots$ it *may* have finite discontinuities. If $f(xy)$ is otherwise regular in R, we shall say it is *simply regular with respect to* x *except on the lines* $x = a_1, \cdots$ or that it is *simply irregular with respect to* x.

5. Let $f(xy)$ be continuous at each point of R except on the lines $x = a_1, \cdots x = a_r$; $y = \alpha_1, \cdots y = \alpha_s$. As in 3, let us suppose that all the points of infinite discontinuity cannot be brought on the lines $x = a_1, \cdots$ Let $f(xy)$ be otherwise regular. We shall say $f(xy)$ is *simply irregular with respect to* x, y, or that it is *simply regular except on the lines* $x = a_1, \cdots y = \alpha_1, \cdots$

6. The lines $x = a_1, \cdots y = \alpha_1, \cdots$ on which are grouped the points of infinite discontinuities of $f(xy)$, are called *singular lines.* To each of these belong right and left hand singular integrals as in Chapter XIV. Cf. 666.

7. The integral
$$\int_x^\infty f(xy)dx, \qquad x \gtreqless G, \tag{1}$$
where G is large at pleasure, is called the *singular integral relative to the line* $x = \infty$.

If for each $\epsilon > 0$ there exists a G, such that 1) is numerically $< \epsilon$ for any y in $\mathfrak{B}$ and every $x \gtreqless G$, we say 1) is *uniformly evanescent in* $\mathfrak{B}$.

8. If the singular integrals relative to the lines $x = a_1, \cdots x = a_r$, as well as the singular integral relative to the line $x = \infty$ are uniformly evanescent in $\mathfrak{B}$, we say the integral
$$J = \int_a^\infty f(x, y)dx \tag{2}$$
is uniformly convergent in $\mathfrak{B}$.

If the uniform convergence of J breaks down at certain points $\gamma_1, \cdots \gamma_t$ in $\mathfrak{B}$, we shall say J is in general uniformly convergent in $\mathfrak{B}$. Cf. 666.

9. As in Chapters XIII, XIV, we wish now to study the integral 2) with respect to continuity, differentiation, and integration. We may often simplify our demonstrations without loss of generality by observing that we may write

$$\int_a^\infty f dx = \int_a^b f dx + \int_b^\infty f dx = J_1 + J_2.$$

Here we may take b so large that none of the lines $x = a_1, \cdots$ fall in $(b \infty \alpha\beta)$.

The integral J_1 has been treated in Chapter XIV.

10. In this article we have considered $f(xy)$ chiefly with respect to x. Evidently we may interchange x and y, which will give us similar definitions with respect to y.

We wish also to note that all the following theorems apply to the integral

$$\int_a^\infty f(x, y) dy,$$

on interchanging x and y.

659. *Let $f(xy)$ be regular in $R = (a \infty \alpha\beta)$, β finite or infinite. Let $\phi(x)$ be integrable in $\mathfrak{A}$, and*

$$|f(xy)| \le \phi(x), \qquad \textit{in } R.$$

Then

$$\int_a^\infty f(x, y) dx \tag{1}$$

is uniformly convergent in $\mathfrak{B}$.

For

$$\left|\int_{b'}^{b''} f dx\right| \le \int_{b'}^{b''} |f| dx, \qquad \text{by 528.}$$

$$\le \int_{b'}^{b''} \phi dx, \qquad \text{by 526, 2.}$$

Since ϕ is integrable in $\mathfrak{A}$, we can take b so large that

$$\int_{b'}^{b''} \phi dx < \epsilon$$

for any pair of numbers $b', b'' > b$.

Hence 1) is uniformly convergent in $\mathfrak{B}$.

660. 1. *Let $f(xy)$ be regular in $R = (a\infty\alpha\beta)$, β finite or infinite. Let*

$$f(x, y) = \phi(x)g(x, y),$$

where, 1°, ϕ is absolutely integrable in $\mathfrak{A}$. 2°, $g(xy)$ is limited in R and integrable in any (a, b), for each y in $\mathfrak{B}$.

Then

$$\int_a^\infty f(xy)dx \tag{1}$$

is uniformly convergent in $\mathfrak{B}$.

For, g being limited in R, let

$$|g(xy)| \leq M.$$

By 1°, there exists for each $\epsilon > 0$, a b such that

$$\int_{b'}^{b''} |\phi(x)| dx < \frac{\epsilon}{M}, \qquad b < b' < b''. \tag{2}$$

Then for any y in $\mathfrak{B}$,

$$\left|\int_{b'}^{b''} f dx\right| = \left|\int_{b'}^{b''} \phi g dx\right|$$

$$\leq M \int_{b'}^{b''} |\phi| dx, \qquad \text{by 529.}$$

$$< \epsilon, \qquad \text{by 2).}$$

Hence 1) is uniformly convergent in $\mathfrak{B}$.

2. As corollary of 1, we have, by 635:

In $R = (a\infty\alpha\beta)$; $a > 0$, β finite or infinite, let

$$f(xy) = \frac{g(x, y)}{x^\lambda}, \text{ or } \frac{x^\mu g(xy)}{e^{\nu x}}; \ \lambda > 1, \ \nu > 0.$$

where g is limited in R, and integrable in any (a, b) for each y in $\mathfrak{B}$. Then

$$\int_a^\infty f(xy)dx$$

is uniformly convergent in $\mathfrak{B}$.

661. 1. *Let* $f(xy)$ *be regular in* $R = (a \infty \alpha\beta)$, β *finite or infinite. Let*

$$f(xy) = \phi(x) g(xy),$$

where 1°, $\phi(x)$ *is integrable in* $\mathfrak{A}$. 2°, $g(xy)$ *is limited in* R *and a monotone function of* x *for any* y *in* $\mathfrak{B}$.

Then

$$\int_a^\infty f(xy)dx$$

is uniformly convergent in $\mathfrak{B}$.

For, by the Second Theorem of the Mean, 545,

$$\int_{b'}^{b''} \phi g dx = g(b'+0, y) \int_{b'}^{\xi} \phi dx + g(b''-0, y) \int_{\xi}^{b''} \phi dx, \quad b < b' < \xi < b''.$$

But g being limited in R, and ϕ integrable, the right side is numerically $< \epsilon$ for any y in $\mathfrak{B}$, and any pair of numbers b', b'', provided b is taken large enough.

2. *Let* $f(xy)$ *be regular in* $R = (a \infty \alpha\beta)$, β *finite or infinite. Let*

$$f(xy) = g(xy) h(xy),$$

where 1°, $h(xy)$ *is limited in* R *and monotone for each* y *in* $\mathfrak{B}$, *and* 2°,

$$\int_a^\infty g(xy)dx$$

is uniformly convergent in $\mathfrak{B}$.

Then

$$\int_a^\infty f(xy)dx$$

is uniformly convergent in $\mathfrak{B}$.

For, by 545,

$$\int_{b'}^{b''} f dx = h(b'+0, y) \int_{b'}^{\xi} g dx + h(b''-0, y) \int_{\xi}^{b''} g dx. \qquad (1$$

But h being limited in R,

$$|h(xy)| < M.$$

On the other hand, by 2°, there exists a b_0 such that each integral on the right of 1) is numerically $< \epsilon/2\,M$ for any pair of numbers $b', b'' > b_0$.

Hence for any y in $\mathfrak{B}$,

$$\left|\int_{b'}^{b''} f dx\right| < \epsilon.$$

662. *Integration by Parts. Let* $f(xy)$ *be regular in* $R=(a\infty\alpha\beta)$, β *finite or infinite. The integral*

$$J=\int_a^\infty f(xy)dx$$

is uniformly convergent in $\mathfrak{B}$, *if*

$$\int_x^\infty f(x, y)dx = F(x, y) + \int_x^\infty g(x, y)dx\,; \tag{1}$$

where both expressions on the right are uniformly evanescent for $x \doteq \infty$.

For then, for each $\epsilon > 0$ there exists a b such that

$$\left|F(x, y)\right| < \frac{\epsilon}{2}, \quad \left|\int_x^\infty g dx\right| < \frac{\epsilon}{2}, \tag{2}$$

for any $x > b$, and any y in $\mathfrak{B}$.

Hence 1) and 2) give

$$\left|\int_x^\infty f dx\right| < \epsilon.$$

663. *Examples.*

1. The integral

$$\Gamma(y) = \int_0^\infty e^{-x} x^{y-1}\, dx, \tag{1}$$

defining the Gamma function, considered in 642, is uniformly convergent in $\mathfrak{B} = (\alpha, \beta)$, $\alpha > 0$.

For, consider the singular integral relative to $x = 0$.

We have, since $0 < x < 1$,

$$x^{y-1} \leq x^{\alpha-1}, \qquad \text{in } \mathfrak{B}.$$

Hence

$$e^{-x}x^{y-1} \leq x^{\alpha-1}.$$

Thus, by 612, the singular integral relative to $x = 0$ is uniformly evanescent in $\mathfrak{B}$. Consider next the singular integral relative to $x = \infty$. We have, since $x > 1$,

$$e^{-x}x^{y-1} \leqq \frac{x^{\beta-1}}{e^x}, \qquad \text{in } \mathfrak{B}.$$

Hence this singular integral is uniformly evanescent in $\mathfrak{B}$. Thus 1) is uniformly convergent in $\mathfrak{B}$.

2.
$$J = \int_1^\infty \frac{\sin xy}{x}\,dx.$$

This is uniformly convergent in any $\mathfrak{B} = (\alpha, \infty)$, which does not contain the point $y = 0$, as may be seen by 662. For, integrating by parts,

$$\int_x^\infty \frac{\sin xy}{x}\,dx = -\left[\frac{\cos xy}{xy}\right]_x^\infty - \int_x^\infty \frac{\cos xy}{x^2y}\,dx$$

$$= \frac{\cos xy}{xy} - \int_x^\infty \frac{\cos xy}{x^2y}\,dx. \tag{2}$$

To fix the ideas, suppose $\alpha > 0$; then

$$\left|\frac{\cos xy}{xy}\right| \leq \frac{1}{\alpha x}.$$

This shows that the first term on the right of 2) is uniformly evanescent in $\mathfrak{B}$. The second term is uniformly evanescent by 660, 1, as is seen, setting

$$\phi(x) = \frac{1}{x^2}, \quad g(xy) = \frac{\cos xy}{y}.$$

For later use, let us note that

$$\lim_{y=\infty} \frac{\cos xy}{xy} = 0, \quad \lim_{y=\infty} \frac{1}{y}\int_x^\infty \frac{\cos xy}{x^2}\,dx = 0.$$

Hence
$$\lim_{y=\infty} J = 0.$$

3.
$$\int_0^\infty \frac{1-e^{-xy}}{x}\sin\lambda x\,dx \tag{3}$$

is uniformly convergent in $\mathfrak{B} = (0, \infty)$, by 661. For, in the first place, the integrand $f(xy)$ is continuous in $R = (0, \infty, 0, \infty)$. For, the only possible points of discontinuity lie on the line $x = 0$.

But, the Law of the Mean gives,

$$e^{-xy} = 1 - xy + \frac{x^2y^2}{2\,!}e^{-\theta xy}.$$

Hence for $x \neq 0$,
$$f(xy) = y\sin\lambda x - xy^2e^{-\theta xy}\sin\lambda x, \qquad 0 < \theta < 1.$$

This shows that f is continuous at each point on the y axis, if we give to f the value 0 at these points.

This fact established, we can apply 661 by setting

$$\phi(x) = \frac{\sin\lambda x}{x}, \quad g(xy) = 1 - e^{-xy}.$$

Then ϕ is integrable in $(0, \infty)$ by 646; while g is obviously limited in R, and a monotone increasing function of x for each y in $\mathfrak{B}$. Hence 3) is uniformly convergent in $\mathfrak{B}$.

4.
$$\int_0^\infty \frac{\sin xy \cos \lambda x}{x}\, dx \tag{4}$$

is uniformly convergent in $\mathfrak{B} = (0, \infty)$ except at $y = |\lambda|$.

For, in the first place the integrand $f(x, y)$ is continuous in $R = (0\infty 0\infty)$ if we give to f the value y at the point $(0, y)$.

For, the Law of the Mean gives

$$\sin xy = xy - \frac{x^2y^2}{2} \sin \theta xy, \qquad 0 < \theta < 1.$$

Hence for $x \neq 0$,

$$f(xy) = y \cos \lambda x - \frac{xy^2}{2} \sin \theta xy \cos \lambda x.$$

Thus

$$\lim_{x=0,\, h=0} f(x, y+h) = y.$$

This established, we have only to show that the singular integral

$$B = \int_{b'}^{b''} f(xy) dx, \qquad b < b' < b'',$$

is uniformly evanescent in $\mathfrak{B}$.

Now by the Second Theorem of the Mean, 545,

$$B = \frac{1}{b'} \int_{b'}^{\xi} \sin xy \cos \lambda x\, dx + \frac{1}{b''} \int_{\xi}^{b''} \sin xy \cos \lambda x\, dx. \tag{5}$$

But for $y \neq |\lambda|$,

$$\int \sin xy \cos \lambda x\, dx = -\frac{\cos (y-\lambda)}{2(y-\lambda)} - \frac{\cos (y+\lambda)}{2(y+\lambda)}. \tag{6}$$

Let

$$|y - \lambda|, \qquad |y + \lambda| > \sigma. \tag{7}$$

Then 6) shows that each of the integrals in 5) is numerically $< 2/\sigma$.

Let therefore, $b > 4/\epsilon\sigma$; then

$$|B| < \epsilon,$$

for any y satisfying 7). That is, B is uniformly evanescent except at $y = |\lambda|$. Hence the integral 4) is uniformly convergent except at this point.

We may arrive at this result more shortly, making use of 2.

For,

$$2 \sin xy \cos \lambda x = \sin x(y+\lambda) + \sin x(y-\lambda).$$

Hence

$$\int_0^\infty \frac{\sin xy \cos \lambda x}{x}\, dx = \int_0^\infty \frac{\sin x(y+\lambda)}{x}\, dx + \int_0^\infty \frac{\sin x(y-\lambda)}{x}\, dx.$$

Here the first integral is uniformly convergent, if $y \neq -\lambda$; the second integral is uniformly convergent, if $y \neq \lambda$.

5.
$$\int_0^\infty \frac{x \sin xy}{1+x^2}\, dx$$

is uniformly convergent in (α, ∞), $\alpha > 0$.

For

$$\frac{x \sin xy}{1+x^2} = \frac{\sin xy}{x} \cdot \frac{1}{1 + \frac{1}{x^2}}.$$

We have now only to apply 661, 2, using the result obtained in Ex. 2.

6.
$$\int_0^\infty \frac{3x^2-y}{3x}\sin(x^3-xy)dx. \tag{8}$$

We assign the value 0 to the integrand, for $x=0$.

To show that this integral is uniformly convergent in any $\mathfrak{B}=(\alpha\beta)$, let us use the method of 662. If we set

$$u=\frac{1}{3x}, \qquad dv=(3x^2-y)\sin(x^3-xy);$$

$$\int_x^\infty udv = \left[-\frac{\cos(x^3-xy)}{3x}\right]_x^\infty - \int_x^\infty \frac{\cos(x^3-xy)}{3x^2}dx$$
$$= \frac{\cos(x^3-xy)}{3x} - \int_x^\infty \frac{\cos(x^3-xy)}{3x^2}dx.$$

Here both terms are uniformly evanescent in $\mathfrak{B}$ by 660, 2. Hence 8) is uniformly convergent in $\mathfrak{B}$.

7.
$$\int_0^\infty \cos(x^3-xy)dx. \tag{9}$$

We can write

$$\int_x^\infty \cos(x^3-xy)dx = \int_x^\infty \frac{3x^2-y}{3x^2}\cos(x^3-xy)dx + \frac{y}{3}\int_x^\infty \frac{\cos(x^3-xy)}{x^2}dx.$$

The second integral on the right is uniformly evanescent by 660, 2. The first integral is also uniformly evanescent. For integrating by parts,

$$\int_x^\infty \frac{3x^2-y}{3x^2}\cos(x^3-xy)dx = -\frac{\sin(x^3-xy)}{3x^2} + \frac{2}{3}\int_x^\infty \frac{\sin(3x^3-xy)}{x^3}dx.$$

Here both terms on the right are obviously uniformly evanescent.

664. Let $f(xy)$ be regular in $R=(a\infty\alpha\beta)$, β finite or infinite. Let

$$J=\int_a^\infty f(xy)dx$$

converge uniformly in $\mathfrak{B}$, except possibly at $\alpha_1, \cdots \alpha_m$. To establish the uniform convergence of J throughout $\mathfrak{B}$, we have only to show that J is uniformly convergent in each of the little intervals

$$\mathfrak{B}_\kappa=(\alpha_\kappa-\delta,\ \alpha_\kappa+\delta).$$

That is, we have only to show that for each $\epsilon>0$, and for some $\delta>0$, there exists a b_κ such that

$$\left|\int_{b'}^\infty f(xy)dx\right|<\epsilon$$

for any y in $\mathfrak{B}_\kappa$ and every $b'>b_\kappa$, $\kappa=1, 2 \cdots m$.

665. *Examples.*

1.
$$J = \int_0^\infty \frac{\sin y \sin xy}{x}\,dx$$

is uniformly convergent in $\mathfrak{B} = (0, \infty)$.

For, in the first place, the integrand $f(xy)$ is continuous in $R = (0, \infty, 0, \infty)$ if we set

$$f(0, y) = y \sin y.$$

We have therefore only to consider the singular integral relative to $x = \infty$, in the intervals $\mathfrak{B}_1 = (0, \delta)$, $\mathfrak{B}_2 = (\delta, \infty)$. Now as in 663, 2, we have for $y > 0$,

$$K = \int_x^\infty \frac{\sin y \sin xy}{x}\,dx = \frac{\sin y \cos xy}{xy} - \sin y \int_x^\infty \frac{\cos xy}{x^2 y}\,dx.$$

The reasoning of 663, 2 shows that K is uniformly evanescent in $\mathfrak{B}_2$.

As to $\mathfrak{B}_1$, we note first that $K = 0$ for $y = 0$. Also that

$$\frac{\sin y}{y} = 1 + \eta', \qquad |\eta'| < \eta,$$

η being as small as we choose, if δ is taken small enough.

Hence for any y in $\mathfrak{B}_1$,

$$|K| < \frac{1+\eta}{x} + (1+\eta)\int_x^\infty \frac{dx}{x^2},$$

which shows that K is uniformly evanescent in $\mathfrak{B}_1$.

2.
$$\int_0^\infty \frac{\sin xy}{ye^{\lambda x}}\,dx, \qquad \lambda > 0 \qquad (1$$

is uniformly convergent in $\mathfrak{B} = (0, \beta)$.

For, the integrand $f(xy)$ is continuous in $R = (0\,\infty\,0\,\beta)$, if we set

$$f(x, 0) = \frac{x}{e^{\lambda x}}.$$

Let us consider therefore the singular integral relative to $x = \infty$. We set $\mathfrak{B}_1 = (0, \delta)$, $\mathfrak{B}_2 = (\delta, \infty)$. Obviously 1) is uniformly convergent in $\mathfrak{B}_2$.

To show the same for $\mathfrak{B}_1$, we note that

$$\sin xy = xy + \tau x^3 y^3, \qquad |\tau| < 1,$$

by the Law of the Mean. Hence

$$|f(xy)| \leq \frac{x}{e^{\lambda x}} + \frac{x^3\delta^2}{e^{\lambda x}}, \qquad \text{in } (0, \infty, 0, \delta).$$

Thus, by 659, the integral 1) is uniformly convergent in $\mathfrak{B}_1$.

Continuity

666. 1. *Let $f(xy)$ be regular in $R = (a\infty\,\alpha\beta)$, β finite or infinite, except on the lines $x = a_1 \cdots x = a_r$.*

1°. *Let*

$$J = \int_a^\infty f(xy)\,dx$$

be uniformly convergent in $\mathfrak{B}$.

2°. *Let*

$$\lim_{y=\eta} f(xy) = \phi(x), \qquad \eta \textit{ finite or infinite}$$

uniformly in any (a, b), except possibly on the lines $x = a_1 \cdots$

Then

$$j = \lim_{y=\eta} \int_a^\infty f(x, y)\,dx \qquad \textit{exists.} \qquad (1$$

3°. *Let $\phi(x)$ be integrable in any (a, b).*

Then

$$\lim_{y=\eta} J = \lim_{y=\eta} \int_a^\infty f(xy)\,dx = \int_a^\infty \phi(x)\,dx. \qquad (2$$

By virtue of 616 we may assume that f is regular in R, and that $f \doteq \phi$ uniformly in any (a, b).

To fix the ideas let $\eta = \infty$.

We show first that j exists; i.e. for each $\epsilon > 0$ there exists a γ such that

$$D = \int_a^\infty \{f(x, y') - f(x, y'')\}\,dx$$

is numerically $\dot{<} \epsilon$ for any pair of numbers $y', y'' > \gamma$.

Now

$$D = \int_a^b \{f(xy') - f(xy'')\}\,dx + \int_b^\infty f(xy')\,dx - \int_b^\infty f(xy'')\,dx$$

$$= D_1 + D_2 + D_3.$$

By 1°, there exists a b such that

$$|D_2|,\ |D_3| < \epsilon/4 \qquad (3$$

for any y', y'' in $\mathfrak{B}$.

By 2°, we can take γ such that for any x in (a, b)

$$|f(xy) - \phi(x)| < \frac{\epsilon}{4(b-a)}, \qquad y > \gamma.$$

Hence

$$|f(x, y') - f(x, y'')| < \frac{\epsilon}{2(b-a)},$$

for any $y', y'' > \gamma$, and x in (a, b).

Thus

$$|D_1| < \epsilon/2. \tag{4}$$

From 3), 4) we have

$$|D| < \epsilon.$$

We next show that 2) *holds; i.e.* for each $\epsilon > 0$ there exists a b_0, such that

$$\left| j - \int_a^b \phi dx \right| < \epsilon \tag{5}$$

for any $b > b_0$.

From 1), there exists a γ such that

$$j = \int_a^\infty f(xy)dx + \epsilon', \qquad |\epsilon'| < \frac{\epsilon}{3} \tag{6}$$

for any $y > \gamma$.

From 1°, there exists a b_0 such that

$$\int_a^\infty f(xy)dx = \int_a^b f(xy)dx + \epsilon'', \qquad |\epsilon''| < \frac{\epsilon}{3} \tag{7}$$

for any $b > b_0$, and any y in $\mathfrak{B}$.

From 2°, we can take γ large enough so that also

$$f(x, y) = \phi(x) + g(xy), \qquad |g| < \frac{\epsilon}{3(b-a)} \tag{8}$$

for any x in (a, b), and any $y > \gamma$.

Hence, by 3°,

$$\int_a^b f(xy)dx = \int_a^b \phi(x)dx + \epsilon''', \qquad |\epsilon'''| < \frac{\epsilon}{3} \tag{9}$$

for any $b > b_0$, and any $y > \gamma$.

From 6), 7), 8), 9) we have

$$j = \int_a^b \phi(x)dx + \epsilon_1 \qquad |\epsilon_1| < \epsilon$$

for any $b > b_0$. But this is 5).

2. The reader should note that the lines $x = a_1, \cdots$ on which the uniform convergence of $f(xy)$ to $\phi(x)$ breaks down, are according to 617, 3 singular lines, whether $f(xy)$ has points of infinite discontinuity on them or not. If the integral J is to be uniformly

convergent in $\mathfrak{B}$, the singular integrals relative to all these lines must be uniformly evanescent.

667. *Example.* As we shall show in 675,

$$J = \int_0^\infty \frac{1-e^{-xy}}{x} \sin \lambda x \, dx = \text{arc tg} \frac{y}{\lambda}, \qquad \lambda \neq 0.$$

The application of 666 gives

$$\lim_{y=\infty} J = \int_0^\infty \frac{\sin \lambda x}{x} dx = \begin{cases} \pi/2, & \lambda > 0. \\ 0, & \lambda = 0. \\ -\pi/2, & \lambda < 0. \end{cases} \tag{1}$$

That the conditions of Theorem 666 are fulfilled is easily seen. For, defining the integrand $f(xy)$ of J as in 663, 3, it is continuous in $R = (0, \infty, 0, \infty)$. We also saw that the singular integral relative to $x = \infty$ is uniformly evanescent in $\mathfrak{B} = (0, \infty)$.

Now

$$\lim_{y=\infty} f(x, y) = \frac{\sin \lambda x}{x}$$

uniformly in $\mathfrak{A} = (0, \infty)$ except at $x = 0$.

The line $x = 0$ is therefore a singular line by 617, 3. But

$$\left| \int_0^a \frac{1-e^{-xy}}{x} \sin \lambda x \, dx \right| \leq \left| \int_0^a \frac{\sin \lambda x}{x} dx \right| < \epsilon, \qquad 0 < a < \delta$$

if δ is taken sufficiently small. This singular integral is therefore uniformly evanescent. Hence J is uniformly convergent in $\mathfrak{B}$. Thus all the conditions of 666 are satisfied.

From 1) we may deduce the following relations:

$$\int_0^\infty \frac{\sin \alpha x \cos \beta x}{x} dx = 0, \tag{2}$$

$$0 \leq \alpha < \beta.$$

$$\int_0^\infty \frac{\cos \alpha x \sin \beta x}{x} dx = \frac{\pi}{2}, \tag{3}$$

which may be combined in a single formula

$$\int_0^\infty \frac{\sin \lambda x \cos \mu x}{x} dx = \begin{cases} 0, & 0 \leq \lambda < \mu. \\ \pi/2, & 0 \leq \mu < \lambda. \end{cases} \tag{4}$$

In fact, since $\alpha + \beta > 0$, $\alpha - \beta < 0$, we have from 1),

$$\int_0^\infty \frac{\sin(\alpha+\beta)x}{x} dx = \frac{\pi}{2}, \tag{5}$$

$$\int_0^\infty \frac{\sin(\alpha-\beta)x}{x} dx = -\frac{\pi}{2}. \tag{6}$$

But

$$\sin(\alpha+\beta)x + \sin(\alpha-\beta)x = 2 \sin \alpha x \cos \beta x, \tag{7}$$

$$\sin(\alpha+\beta)x - \sin(\alpha-\beta)x = 2 \cos \alpha x \sin \beta x. \tag{8}$$

Adding and subtracting 5), 6) and using 7), 8), we get 2), 3).

668. That the relation

$$\lim_{y=\eta}\int_a^\infty f(x,\,y)dx=\int_a^\infty \lim_{y=\eta} f(x,\,y)dx \qquad (1$$

is not always true is shown by the following example:

From

$$\int e^{-xy}\sin x\,dx=-\frac{e^{-xy}(\cos x+y\sin x)}{1+y^2},$$

we have for $y>0$,

$$\int_0^\infty \frac{\sin x}{e^{xy}}dx=\frac{1}{1+y^2}. \qquad (2$$

$$R\lim_{y=0}\int_0^\infty \frac{\sin x}{e^{xy}}dx=1.$$

On the other hand,

$$\int_0^\infty R\lim_{y=0}\frac{\sin x}{e^{xy}}dx=\int_0^\infty \sin x\,dx$$

does not even exist.

Thus the relation 1) does not hold.

669. 1. *Let* $f(xy)$ *be regular in* $R=(a\infty\alpha\beta)$ *except possibly on the lines* $x=a_1, \cdots x=a_r$.

Let $f(xy)$ *be a uniformly continuous function of* y *in* $\mathfrak{B}$ *except possibly on the lines* $x=a_1, \cdots$ *Let*

$$J(y)=\int_a^\infty f(xy)dx$$

be uniformly convergent in $\mathfrak{B}$.

Then J *is continuous in* $\mathfrak{B}$.

For, $f(x,\,y+h)$ converges uniformly to $f(x,\,y)$, $h\doteq 0$ in $\mathfrak{B}$ except on the lines $x=a_1, \cdots$ We have, therefore, only to apply 666.

2. *Let* $f(xy)$ *be in general regular with respect to* x *in* $R=(a\infty\alpha\beta)$. *Let* f *be in general a semi-uniformly continuous function of* y *in* $\mathfrak{B}$. *Let*

$$J(y)=\int_a^\infty f(xy)dx$$

be uniformly convergent in $\mathfrak{B}$.

Then J *is limited in* $\mathfrak{B}$, *and in general a continuous function of* y.

For, we can take b so large that

$$\left|\int_b^\infty f dx\right| < \epsilon$$

for any y in $\mathfrak{B}$. On the other hand,

$$\int_a^b f dx$$

is limited in $\mathfrak{B}$ by 618, 1. Hence J is limited in $\mathfrak{B}$. That J is in general continuous in $\mathfrak{B}$ follows from 1.

3. In this connection let us note the following theorem whose demonstration is obvious.

Let $f(xy)$ *be regular in* $R = (a \infty \alpha\beta)$, β *finite or infinite, except on the lines* $x = a_1, \cdots;\ y = \alpha_1, \cdots$ *Let*

$$\int_\alpha^\beta f(xy) dy$$

be uniformly convergent in any (a, b) *except at* $a_1, a_2 \cdots$ *Then the points of infinite discontinuity of*

$$g(xy) = \int_\alpha^y f(xy) dy, \qquad y \text{ in } \mathfrak{B}.$$

must lie on the lines $x = a_1, \cdots$

670. 1. *Let* $f(xy)$ *be regular in* $R = (a \infty \alpha\beta)$, *except on* $x = a_1 \cdots;$ $y = \alpha_1 \cdots$

1°. *Let*

$$\int_\alpha^\beta f dy$$

converge uniformly in any (a, b) *except at* $x = a_1 \cdots$

2°. *Let*

$$\phi(y) = \int_a^\infty dx \int_\alpha^y f dy$$

converge uniformly in $\mathfrak{B}$.

Then ϕ *is continuous in* $\mathfrak{B}$.

This is a direct application of 666, 1, where

$$g(xy) = \int_\alpha^y f(xy) dy$$

takes the place of f in that theorem.

In fact, by 669, 3, $g(xy)$ has no points of infinite discontinuity except on $x = a_1 \cdots$; and is therefore by 2°, regular in R except on these lines.

Also $g(x, y+h)$ converges uniformly in any (a, b) to $g(x, y)$ as $h \doteq 0$, except at $x = a_1 \cdots$; since

$$g(x, y+h) - g(xy) = \int_y^{y+h} f dy$$

is uniformly evanescent in (a, b) by 1°.

Thus applying 666, 1, we have

$$\lim_{h=0} \phi(y+h) = \lim_{h=0} \int_a^\infty g(x, y+h)dx = \int_a^\infty \lim_{h=0} g(x, y+h)dx$$

$$= \int_a^\infty dx \int_a^y f dy = \phi(y).$$

That is, $\phi(y)$ is continuous at y.

2. As a corollary of 1 we have:

Let $f(xy)$ *be in general regular with respect to* x *in* $R = (a\infty\alpha\beta)$. *Let*

$$\phi(y) = \int_a^\infty dx \int_a^y f(xy)dy$$

converge uniformly in $\mathfrak{B}$. *Then* ϕ *is continuous in* $\mathfrak{B}$.

Integration and Inversion

671. 1. *Let* $f(xy)$ *be in general regular with respect to* x *in* $R = (a\infty\alpha\beta)$. *Let* f *be in general a semi-uniformly continuous function of* y *in* $\mathfrak{B}$. *Let*

$$J(y) = \int_a^\infty f(xy)dx$$

be uniformly convergent in $\mathfrak{B}$.

Then J *is integrable in* $\mathfrak{B}$.

This follows at once from 500 and 669, 2.

2. As a corollary of 1 we have:

Let $f(xy)$ *be simply irregular with respect to* x *in* $R = (a\infty\alpha\beta)$. *Let* J *be uniformly convergent in* $\mathfrak{B}$. *Then* J *is integrable in* $\mathfrak{B}$.

672. 1. *Let $f(xy)$ be in general regular in $R = (a\infty\alpha\beta)$, with respect to x.*

1°. *Let*

$$\int_a^\infty f dx \tag{1}$$

be uniformly convergent, and integrable in $\mathfrak{B}$.

2°. *Let*

$$\int_a^y dy \int_a^b f dx, \qquad b \text{ arbitrarily large,}$$

admit inversion in $\mathfrak{B}$.

Then

$$\int_a^y dy \int_a^\infty f dx = \int_a^\infty dx \int_a^y f dy, \qquad \text{in } \mathfrak{B}.$$

We set

$$\int_a^\infty f dx = \int_a^b + \int_b^\infty . \tag{2}$$

Since 1) is uniformly convergent in $\mathfrak{B}$, there exists for each $\epsilon > 0$, a b_0 such that

$$\left| \int_b^\infty f dx \right| < \frac{\epsilon}{\beta - \alpha}, \tag{3}$$

for any y in $\mathfrak{B}$, and every $b > b_0$.

Thus 2), 3) give for any y in $\mathfrak{B}$,

$$\left| \int_a^y dy \int_a^\infty f dx - \int_a^y dy \int_a^b f dx \right| \leq \int_a^y \frac{\epsilon dx}{\beta - \alpha} \leqq \epsilon. \tag{4}$$

But by 2°,

$$\int_a^y \int_a^b = \int_a^b \int_a^y .$$

Hence

$$\left| \int_a^y \int_a^\infty - \int_a^b \int_a^y \right| \leq \epsilon, \qquad b > b_0, \tag{5}$$

which proves the theorem.

2. *Let $f(xy)$ be simply irregular with respect to x in $R = (a\infty\alpha\beta)$. Let*

$$\int_a^\infty f dx$$

be uniformly convergent in $\mathfrak{B}$.

Then

$$\int_a^\beta dy \int_a^\infty f dx, \quad \int_a^\infty dx \int_a^\beta f dy \tag{6}$$

are convergent and equal.

For, by 671, 2, the integral on the left of 6) exists. Moreover, condition 2° of 1 is fulfilled, by 622, 2.

3. As corollary of 1, we have:

Let $f(xy)$ be in general regular in $R = (a\infty\alpha\beta)$ with respect to x. Let

$$\int_a^\infty f dx$$

be uniformly convergent, and integrable in $\mathfrak{B}$.

Let

$$\int_a^y dy \int_a^b f dx, \qquad b \text{ arbitrarily large,}$$

admit inversion in $\mathfrak{B}$.

Then

$$\int_a^\infty dx \int_a^y f(xy) dy$$

is uniformly convergent in $\mathfrak{B}$.

This follows at once from 5), since this inequality holds for any y in $\mathfrak{B}$.

4. From the relation 4) we have also the following corollary, setting $y = \beta$.

Let $f(xy)$ be in general regular in $R = (a\infty\alpha\beta)$ with respect to x. Let

$$\int_a^\infty f dx$$

be uniformly convergent, and integrable in $\mathfrak{B}$. Let

$$\int_a^\beta dy \int_a^b f dx, \qquad b \textit{ arbitrarily large,}$$

admit inversion. Then

$$\lim_{b=\infty} \int_a^\beta dy \int_a^b f dx = \int_a^\beta dy \int_a^\infty f dx.$$

5. As a special case of 4 we have, 622, 2 and 671, 2:

Let $f(xy)$ *be simply irregular with respect to* x *in* $R = (a\infty\alpha\beta)$. *Let*

$$\int_a^\infty f dx$$

be uniformly convergent in $\mathfrak{B}$. *Then*

$$\lim_{b=\infty} \int_\alpha^\beta dy \int_a^b f dx = \int_\alpha^\beta dy \int_a^\infty f dx.$$

673. *Let* $f(xy)$ *be regular in* $R = (a\infty\alpha\beta)$, *except on the lines*, $x = a_1, \cdots x = a_r;\ y = \alpha_1, \cdots y = \alpha_s$.

1°. *Let*

$$\int_\lambda^\mu dy \int_a^\infty f dx$$

be convergent, and admit inversion in any interval (λ, μ), *which does not embrace* $\alpha_1 \cdots \alpha_s$.

2°. *Let*

$$\int_a^\infty dx \int_\alpha^y f dy$$

be a continuous function of y *in* $\mathfrak{B}$.

Let

$$K = \int_\alpha^\beta dy \int_a^\infty f dx, \qquad L = \int_a^\infty dx \int_\alpha^\beta f dy.$$

Then K *is convergent, and* $K = L$.

For simplicity, let $y = \gamma$ be the only singular y-line; $\alpha < \gamma < \beta$. Then by definition,

$$\int_\alpha^\beta dy \int_a^\infty f dx = \lim_{u=\gamma} \int_\alpha^u \int_a^\infty + \lim_{v=\gamma} \int_v^\beta \int_a^\infty. \tag{1}$$

$$\alpha < u < \gamma, \qquad \gamma < v < \beta.$$

Now

$$\int_\alpha^u \int_a^\infty = \int_a^\infty \int_\alpha^u, \quad \text{by 1°.} \tag{2}$$

Similarly,

$$\int_v^\beta \int_a^\infty = \int_a^\infty \int_v^\beta = \int_a^\infty \int_\alpha^\beta - \int_a^\infty \int_\alpha^v, \tag{3}$$

since L is by 2°, convergent.

Now by 2°,

$$\lim \int_a^\infty \int_\alpha^u = \int_a^\infty \int_\alpha^\gamma ; \tag{4}$$

$$\lim \int_a^\infty \int_\alpha^v = \int_a^\infty \int_\alpha^\gamma . \tag{5}$$

Hence from 2), 4),

$$\lim \int_a^u \int_\alpha^\infty = \int_a^\infty \int_\alpha^\gamma ; \tag{6}$$

also from 3), 5)

$$\lim \int_v^\beta \int_a^\infty = \int_a^\infty \int_\alpha^\beta - \int_a^\infty \int_\alpha^\gamma = \int_a^\infty \int_\gamma^\beta . \tag{7}$$

Hence from 1), 6), 7) we have

$$\int_\alpha^\beta \int_a^\infty = \int_a^\infty \int_\alpha^\gamma + \int_a^\infty \int_\gamma^\beta = \int_a^\infty \int_\alpha^\beta .$$

674. 1. *Let* $f(xy)$ *be simply regular in* $R = (a \infty \alpha \beta)$, *except on the lines* $x = a_1, \cdots y = \alpha_1 \cdots$

1°. *Let*

$$\int_a^\infty f(xy)dx$$

converge uniformly in $\mathfrak{B}$, *except on* $y = \alpha_1 \cdots$

2°. *Let*

$$\int_\alpha^\beta f(xy)dy$$

converge uniformly in any (a, b), *except on* $x = a_1 \cdots$

3°. *Let*

$$\int_a^\infty dx \int_\alpha^y f(xy)dy$$

converge uniformly in $\mathfrak{B}$.

Let

$$K = \int_\alpha^\beta dy \int_a^\infty f dx, \qquad L = \int_a^\infty dx \int_\alpha^\beta f dy.$$

Then K *is convergent and* $K = L$.

This follows from 673. For, in the first place, condition 1° of 673 is satisfied, by 672, 2.

Secondly, condition 2° of 673 is fulfilled, by 670, 1.

2. As a corollary of 1, we have:

Let $f(xy)$ be simply irregular with respect to x in $R = (a\infty\alpha\beta)$ Let

$$\int_a^\infty f(xy)dx$$

converge in general uniformly in $\mathfrak{B}$. Let

$$\int_a^\infty dx\int_a^y f(xy)dy$$

converge uniformly in $\mathfrak{B}$. Then

$$\int_a^\beta dy\int_a^\infty fdx = \int_a^\infty dx\int_a^\beta fdy.$$

675. For $y > 0$, we have from 668, 2),

$$\int_0^\infty \frac{\sin \lambda x}{e^{xy}}\, dx = \frac{\lambda}{\lambda^2 + y^2}. \tag{1}$$

The integral on the left does not exist for $y = 0$. Let us therefore set

$$f(xy) = \frac{\sin \lambda x}{e^{xy}}, \qquad y > 0;$$

$$= 0, \qquad y = 0.$$

Then integrating 1), from 0 to y we get

$$\int_0^y dy \int_0^\infty f dx = \int_0^y \frac{\lambda}{\lambda^2 + y^2}\, dy = \operatorname{arc\,tg} \frac{y}{\lambda}, \qquad \lambda \neq 0. \tag{2}$$

We may invert the order of integration in 2) by 674, 2. For, f is continuous in $R = (0\infty 0y)$, except on the line $y = 0$, and limited in R. It is therefore simply irregular with respect to x. The integral 1) obviously converges uniformly in $\mathfrak{B}$ except at $y = 0$. The integral

$$\int_0^\infty dx \int_0^y f dy = \int_0^\infty \frac{1 - e^{-xy}}{x} \sin \lambda x\, dx \tag{3}$$

is uniformly convergent in $\mathfrak{B}$ by 663, 3. Hence the integrals on the left in 2), 3) are equal, and

$$\int_0^\infty \frac{1 - e^{-xy}}{x} \sin \lambda x\, dx = \operatorname{arc\,tg} \frac{y}{\lambda}, \qquad \lambda \neq 0.$$

676. We saw in 667 that

$$\int_0^\infty \frac{\sin xy}{x}\, dx = \begin{cases} 0, & y = 0, \\ \dfrac{\pi}{2}, & y > 0. \end{cases} \tag{1}$$

Hence, integrating between 0 and 1, we get

$$\int_0^1 dy \int_0^\infty \frac{\sin xy}{x}\, dx = \frac{\pi}{2}. \tag{2}$$

We can invert the order of integration by 674, 2. For, in the first place, the integrand, not being defined in 2), we can make it continuous in $R = (0\infty 01)$, giving it the value y at the points $(0, y)$. Secondly, the integral 1) is uniformly convergent in $\mathfrak{B}$, except at $y = 0$, by 663, 2. Finally,

$$\int_0^\infty dx \int_0^y \frac{\sin xy}{x}\,dy = \int_0^\infty \frac{1 - \cos xy}{x^2}\,dx$$

is uniformly convergent in $\mathfrak{B}$, since

$$\left|\frac{1 - \cos xy}{x^2}\right| \le \frac{2}{x^2}.$$

We can therefore invert in 2), which gives

$$\frac{\pi}{2} = \int_0^\infty \frac{dx}{x} \int_0^1 \sin xy\,dy = \int_0^\infty \frac{1 - \cos x}{x^2}\,dx$$

$$= 2\int_0^\infty \frac{\sin^2 x/2}{x^2}\,dx = \int_0^\infty \frac{\sin^2 u\,du}{u^2}, \tag{3}$$

setting $x = 2u$.

Thus 3) gives

$$\int_0^\infty \frac{\sin^2 x\,dx}{x^2} = \frac{\pi}{2}. \tag{4}$$

677. That the order of integration can not always be inverted is shown by the following examples.

Ex. 1. Let us consider

$$\int_0^\infty dx \int_0^\beta \cos xy\,dy = \int_0^\infty \frac{\sin \beta x}{x}\,dx \tag{1}$$

$$= \frac{\pi}{2}.$$

The integral obtained by inverting the order of integration, viz.,

$$\int_0^\beta dy \int_0^\infty \cos xy\,dx$$

does not exist, since

$$\int_0^\infty \cos xy\,dx$$

does not. Inversion in the order of integration in 1) is therefore not permissible.

Ex. 2. Let

$$\frac{x^2y^2}{1 + x^4y^4} = \frac{u^2}{1 + u^4} = \phi(u), \qquad u = xy.$$

Then

$$\frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial u}\cdot\frac{\partial u}{\partial x} = y\phi'(u); \qquad \frac{\partial \phi}{\partial y} = x\phi'(u).$$

Let

$$f(x, y) = \phi'(u) = \frac{1}{y}\frac{\partial \phi}{\partial x} = \frac{1}{x}\frac{\partial \phi}{\partial y}.$$

We have also

$$\phi(u) = \tfrac{1}{2} u \cdot \psi'(u),$$

where

$$\psi(u) = \operatorname{arc\,tg} u^2.$$

Thus

$$\frac{\partial\psi}{\partial x} = \frac{\partial\psi}{\partial u}\frac{\partial u}{\partial x} = y\psi'(u),$$

and hence

$$2\frac{\phi(u)}{x} = \frac{\partial\psi}{\partial x}.$$

Hence

$$\int_0^\infty dx \int_0^y f(x, y)dy = \int_0^\infty \frac{dx}{x}\int_0^y \frac{\partial\phi}{\partial y}\,dy = \int_0^\infty \frac{\phi(u)}{x}\,dx$$

$$= \frac{1}{2}\int_0^\infty \frac{\partial\psi}{\partial x}\,dx = \frac{1}{2}\Big[\psi(u)\Big]_0^\infty = \frac{\pi}{4}.$$

On the other hand

$$\int_0^y dy \int_0^\infty f(x, y)dx = \int_0^y \frac{dy}{y}\int_0^\infty \frac{\partial\phi}{\partial x}\,dx = \int_0^y \frac{dy}{y}\Big[\phi(u)\Big]_0^\infty = 0.$$

Thus it is not permissible to invert the order of integration in

$$\int_0^\infty dx \int_0^y f(x, y)dy = \int_0^\infty dx \int_0^y \frac{\partial}{\partial y}\frac{xy^2}{1 + x^4y^4}\,dy. \qquad (2$$

678. 1. *Let $f(xy)$ be regular in $R = (a\infty\,\alpha\infty)$, except on the lines* $x = a_1, \cdots;\ y = \alpha_1, \cdots$

1°. *Let*

$$\int_a^\infty dx \int_\alpha^y f dy$$

be uniformly convergent in $\mathfrak{B}$.

2°. *Let*

$$\phi(x) = \int_\alpha^\infty f dy$$

be uniformly convergent in any (a, b), *except at* $a_1, a_2, \cdots$; *and integrable in* (a, b). *Then*

$$\int_a^\infty dx \int_\alpha^\infty f dy \qquad \textit{exists,}$$

and

$$\lim_{y=\infty}\int_a^\infty dx \int_\alpha^y f dy = \int_a^\infty dx \int_\alpha^\infty f dy. \qquad (1$$

This is a direct application of 666, 1; the function

$$g(xy) = \int_\alpha^y f(xy)dy$$

taking the place of $f(xy)$ in that theorem. For, in the first place, g has no points of infinite discontinuity, except on the lines $x = a_1,$

$\cdots$, by 669, 3. Moreover, $g(xy)$ is integrable in $\mathfrak{A}$, by 1°. Hence $g(xy)$ is regular in R, except on the lines $x = a_1, \cdots$

Secondly,

$$\int_a^\infty g(xy)dx$$

is uniformly convergent in $\mathfrak{B}$, by 1°.

Finally

$$\lim_{y=\infty} g(xy) = \int_a^\infty f dy = \phi(x),$$

uniformly in any (a, b), except on the lines $x = a_1, \cdots$; moreover ϕ is integrable in (a, b). Thus all the conditions of 666, 1 are satisfied, and the present theorem is established.

2. As a corollary of 1 we have:

Let $f(xy)$ *be simply irregular with respect to* y *in* $R = (a\infty a\infty)$. *Let*

$$\int_a^\infty dx \int_a^y f dy$$

be uniformly convergent in $\mathfrak{B}$. *Let*

$$\int_a^\infty f dy \tag{2}$$

be uniformly convergent in any (a, b). *Then*

$$\lim_{y=\infty} \int_a^\infty dx \int_a^y f dy = \int_a^\infty dx \int_a^\infty f dy.$$

For 2) is integrable in any (a, b), by 671, 2; on interchanging x and y in that theorem.

679. If the conditions of 678 are not satisfied, the relation

$$\lim_{y=\infty} \int_a^\infty dx \int_a^y f dy = \int_a^\infty dx \int_a^\infty f dy \tag{1}$$

may be untrue.

Consider, for example,

$$J = \int_a^\infty dx \int_0^y \cos xy dy, \qquad a > 0.$$

$$= \int_a^\infty \frac{\sin xy}{x} dx.$$

Here $$\lim_{y=\infty} J = 0, \qquad \text{by 663, Ex. 2.}$$

On the other hand, the integral

$$\int_a^\infty dx \int_a^\infty \cos xy\, dy$$

does not even exist, since

$$\int_a^\infty \cos xy\, dy$$

does not. Thus the relation 1) in this case is not true.

680. 1. *Let $f(xy)$ be simply regular in $R=(a\infty\alpha\infty)$, except on the lines $x=a_1, \cdots;$ $y=\alpha_1, \cdots$*

1°. *Let*

$$\int_a^\infty f dx$$

be uniformly convergent in any (α, β) except at $\alpha_1, \cdots$

2°. *Let*

$$\int_a^\infty f dy$$

be uniformly convergent in any (a, b) except at $a_1, \cdots;$ moreover let it be integrable in (a, b).

3°. *Let*

$$\int_a^\infty dx \int_a^y f dy$$

be uniformly convergent in $\mathfrak{B}$.

Then

$$\int_a^\infty dy \int_a^\infty f dx, \qquad \int_a^\infty dx \int_a^\infty f dy$$

are convergent and equal.

For, by 674, 1,

$$\int_a^\beta dy \int_a^\infty f dx = \int_a^\infty dx \int_a^\beta f dy.$$

But by 678, 1, we may pass to the limit $\beta=\infty$, which proves the theorem.

2. *Let $f(xy)$ be simply regular with respect to y in $R=(a\infty\alpha\infty)$, except on the lines $y=\alpha_1, \cdots$*

Let

$$\int_a^\infty f dx$$

be uniformly convergent in any (α, β) *except at* $\alpha_1, \cdots$

Let

$$\int_a^\infty f dy$$

be uniformly convergent in any (a, b).

Let

$$\int_a^\infty dx \int_a^y f dy$$

be uniformly convergent in $\mathfrak{B}$.

Then

$$\int_a^\infty dy \int_a^\infty f dx, \qquad \int_a^\infty dx \int_a^\infty f dy$$

are convergent, and equal

This follows as in 1, by 674, 1, and 678, 2.

3. *Let* $f(x, y) \geq 0$ *be simply regular in* $R = (a\infty\alpha\infty)$ *except on the lines* $x = a_1, \cdots; \; y = \alpha_1, \cdots$

Let

$$\int_a^\infty f dx$$

be uniformly convergent in any (α, β) *except at* $\alpha_1, \cdots$

Let

$$\int_a^\infty f dy$$

be uniformly convergent in any (a, b) *except at* $a_1, \cdots$

Let

$$L = \int_a^\infty dx \int_a^\infty f dy$$

be convergent. Then

$$K = \int_a^\infty dy \int_a^\infty f dx$$

is convergent, and $K = L$.

This is a corollary of 1. For, condition 2° is satisfied since L exists. That condition 3° is fulfilled follows from the fact that the singular integrals of

$$\int_a^\infty \int_a^y$$

are $\leq$ the corresponding integrals of L since $f \gtreqless 0$.

681. 1 We saw in 667, 1) that

$$J=\int_0^\infty \frac{\sin xy\cos\lambda x}{x}\,dx=\begin{cases}0, & y<\lambda.\\ \pi/2, & y>\lambda.\end{cases}\qquad \lambda,\ y\gtreqless 0.$$

Multiply by $e^{-\mu y}$, and integrate, $\mu>0$. Then

$$K=\int_0^\infty dy\int_0^\infty \frac{\sin xy\cos\lambda x}{xe^{\mu y}}\,dx=\int_0^\infty Je^{-\mu y}dy=\int_0^\lambda+\int_\lambda^\infty \tag{1}$$

$$=\frac{\pi}{2}\int_\lambda^\infty e^{-\mu y}dy$$

$$=\frac{\pi}{2\,\mu e^{\lambda\mu}}.$$

We can invert the order of integration in 1) by 680, 2. For, in the first place

$$f(xy)=\frac{\sin xy\cos\lambda x}{xe^{\mu y}}$$

is simply regular in $R=(0\infty 0\infty)$, if we set

$$f(0,\ y)=\frac{y}{e^{\mu y}}.$$

Secondly,

$$\int_0^\infty f dx=e^{-\mu y}\int_0^\infty \frac{\sin xy\cos\lambda x}{x}\,dx$$

is uniformly convergent in $\mathfrak{B}=(0,\ \infty)$ except for $y=\lambda$ by 663, Ex. 4.

Thirdly,

$$\int_0^\infty f dy=\cos\lambda x\int_0^\infty \frac{\sin xy}{xe^{\mu y}}\,dy\qquad \text{for } x>0$$

$$=\int_0^\infty \frac{y}{e^{\mu y}}\,dy\qquad \text{for } x=0$$

is uniformly convergent in any $(0,\ b)$ by 665, Ex. 2.

Finally,

$$Y=\int_0^\infty dx\int_0^y f dy=\int_0^\infty dx\int_0^y \frac{\sin xy\cos\lambda x}{xe^{\mu y}}\,dy$$

is uniformly convergent in $\mathfrak{B}$.

For,

$$\int_0^y \frac{\sin xy}{e^{\mu y}}\,dy=\left[-\,e^{-\mu y}\frac{\mu\sin xy+x\cos xy}{\mu^2+x^2}\right]_0^y.$$

Hence

$$Y=\int_0^\infty \frac{\cos\lambda x}{\mu^2+x^2}(1-e^{-\mu y}\cos xy)dx-\mu e^{-\mu y}\int_0^\infty \frac{\sin xy}{x}\cdot\frac{\cos\lambda x}{\mu^2+x^2}\,dx$$

$$=Y_1+Y_2.$$

Y_1, Y_2 are uniformly convergent in $\mathfrak{B}$ by 659. For,

$$\left|\frac{\cos\lambda x}{\mu^2+x^2}(1-e^{-\mu y}\cos xy)\right| \leq \frac{2}{\mu^2+x^2},$$

$$\left|\frac{\sin xy}{x}\,\frac{\cos\lambda x}{\mu^2+x^2}\right| \leq \frac{1}{\mu^2+x^2}.$$

Thus all the conditions of 680, 2 are satisfied.
Inverting therefore in 1), we get

$$K = \int_0^\infty \frac{\cos\lambda x}{x}\,dx \int_0^\infty \frac{\sin xy}{e^{\mu y}}\,dy$$

$$= \int_0^\infty \frac{\cos\lambda x}{x}\,dx \left[-e^{-\mu y}\,\frac{\mu\sin xy + x\cos xy}{\mu^2+x^2}\right]_0^\infty$$

$$= \int_0^\infty \frac{\cos\lambda x}{\mu^2+x^2}\,dx.$$

Comparing with 1), we get

$$\int_0^\infty \frac{\cos\lambda x}{\mu^2+x^2}\,dx = \frac{\pi}{2\,\mu e^{\lambda\mu}}. \qquad \lambda \gtreqless 0,\ \mu > 0. \tag{2}$$

2. Let us integrate 2) with respect to λ. We get

$$\int_0^\lambda d\lambda \int_0^\infty \frac{\cos\lambda x}{\mu^2+x^2}\,dx = \frac{\pi}{2\mu}\int_0^\lambda e^{-\lambda\mu}d\lambda$$

$$= \frac{\pi}{2\mu^2}(1-e^{-\lambda\mu}). \tag{3}$$

We can invert the order of integration in the integral on the left, by 674, 2. For,

$$\int_0^\infty \frac{\cos\lambda x}{\mu^2+x^2}\,dx$$

is uniformly convergent by 659, since

$$\left|\frac{\cos\lambda x}{\mu^2+x^2}\right| \leq \frac{1}{\mu^2+x^2}.$$

In the second place,

$$\int_0^\infty \frac{dx}{\mu^2+x^2}\int_0^\lambda \cos\lambda x\,d\lambda = \int_0^\infty \frac{\sin\lambda x}{x(\mu^2+x^2)}\,dx$$

converges uniformly in any interval $(0, \beta)$ by 661, 2 and 663, Ex. 2.
Inverting therefore in 3), we get

$$\int_0^\infty \frac{\sin xy}{x(\mu^2+x^2)}\,dx = \frac{\pi}{2\mu^2}(1-e^{-\mu y}). \qquad \mu > 0,\ y \gtreqless 0.$$

682. Let us evaluate

$$J = \int_0^\infty \frac{du}{e^{u^2}}, \qquad (1$$

which is convergent by 635, 3.

We change the variable, setting

$$u = xy, \qquad y > 0. \qquad (2$$

Then

$$J = \int_0^\infty \frac{y\,dx}{e^{x^2y^2}}.$$

Multiplying by e^{-y^2} and integrating, we get

$$J\int_a^\infty \frac{dy}{e^{y^2}} = \int_a^\infty dy \int_0^\infty \frac{y\,dx}{e^{y^2(1+x^2)}}.$$

This relation is true for any $\alpha > 0$, by 2).

Passing to the limit $\alpha = 0$, we have, since the limits exist,

$$J \cdot \int_0^\infty \frac{dy}{e^{y^2}} = J^2 = \int_0^\infty dy \int_0^\infty \frac{y\,dx}{e^{y^2(1+x^2)}}. \qquad (3$$

We may invert the order of integration in the integral on the right by 680, 3. For, in the first place, the integrand is regular and continuous in $R = (0\infty 0\infty)$. Secondly,

$$\int_0^\infty \frac{y\,dx}{e^{y^2(1+x^2)}}$$

is uniformly convergent in any (α, β), $\alpha > 0$ by 659, since

$$\frac{y}{e^{y^2(1+x^2)}} < \frac{\beta}{e^{\alpha^2(1+x^2)}}.$$

Thirdly,

$$\int_0^\infty \frac{y\,dy}{e^{y^2(1+x^2)}}$$

is uniformly convergent in $\mathfrak{A} = (0\infty)$ by 659, since

$$\frac{y}{e^{y^2(1+x^2)}} \leq \frac{y}{e^{y^2}}.$$

Finally,

$$L = \int_0^\infty dx \int_0^\infty \frac{y\,dy}{e^{y^2(1+x^2)}} = \int_0^\infty dx \left[-\frac{e^{-y^2(1+x^2)}}{2(1+x^2)} \right]_0^\infty$$

$$= \frac{1}{2}\int_0^\infty \frac{dx}{1+x^2} = \frac{\pi}{4}$$

is convergent. Thus all the conditions of 680, 3 being fulfilled, we can invert in 3), which gives

$$J^2 = L = \pi/4.$$

Hence

$$J = \pm\sqrt{\pi}/2.$$

Here we must take the positive sign, since the integral 1) is positive by 649, 3. Hence, finally,

$$\int_0^\infty \frac{dx}{e^{x^2}} = \frac{\sqrt{\pi}}{2}. \qquad (4$$

Differentiation

683. 1. *Let $f(xy)$, $f'_y(xy)$ be in general regular with respect to x, y in $R = (a \infty \alpha\beta)$.*

1°. *For each x in $\mathfrak{A}$ let f be continuous in y, while f'_y is in general continuous in y.*

2°. *For each y in $\mathfrak{B}$, let*

$$\int_a^b dx \int_\alpha^y f'_y dy, \qquad b \text{ arbitrarily large,}$$

admit inversion. Then

$$\frac{d}{dy}\int_a^\infty f(xy)dx = \frac{d}{dy}\lim_{b=\infty}\int_\alpha^y dy \int_a^b f'_y dx, \tag{1}$$

provided the derivative on either side exists.

For, by 605,

$$\int_\alpha^y f'_y dy = f(x, y) - f(x, \alpha).$$

Hence

$$\int_a^b f dx = \int_a^b dx \int_\alpha^y f'_y dy + \int_a^b f(x, \alpha)dx$$

$$= \int_\alpha^y dy \int_a^b f'_y dx + \int_a^b f(x, \alpha)dx;$$

and therefore

$$\int_a^\infty f dx = \lim_{b=\infty}\int_\alpha^y dy \int_a^b f'_y dx + \int_a^\infty f(x, \alpha)dx.$$

Differentiating, we get 1), since the last term on the right is a constant.

2. As corollary of 1 we have:

Let $f(xy)$ be regular in $R = (a \infty \alpha\beta)$ except on the lines $x = a_1, \cdots$ and continuous with respect to y for each x in $\mathfrak{A}$.

Let f'_y be regular in R except on the lines $x = a_1, \cdots$ and uniformly continuous in y, except on these lines.

Let

$$\int_a^\infty f'_y dx$$

be uniformly convergent in $\mathfrak{B}$.

Then

$$\frac{d}{dy}\int_a^\infty f(xy)dx = \int_a^\infty f'_y dx.$$

For, condition 2° of 1 is fulfilled by 622, 2.

Hence by 1),

$$\frac{d}{dy}\int_a^\infty f(xy)dx = \frac{d}{dy}\lim_{b=\infty}\int_a^y dy\int_a^b f'_y dx$$

$$= \frac{d}{dy}\int_a^y dy \int_a^\infty f'_y dx, \qquad \text{by 672, 4, and 671, 2,}$$

$$= \int_a^\infty f'_y dx, \qquad \text{by 669, 1.}$$

684. 1. When

$$\int_a^\infty f'_y dx$$

is not convergent, the following theorem may serve.

Let $f(xy)$ be in general regular in $R = (a\infty\alpha\beta)$, and continuous with respect to y for each x in $\mathfrak{A}$.

Let f'_y be simply irregular with respect to x in R.

1°. *Let*

$$\int_a^b f'_y dx, \qquad b \text{ arbitrarily large,}$$

be uniformly convergent in $\mathfrak{B}$.

2°. *For any b, let*

$$\int_a^b f'_y dx = \int_a^b g(xy)dx + h(b, y), \qquad \textit{where}$$

3°. *$g(xy)$ is simply irregular in R with respect to x and*

$$\int_a^\infty g(xy)dx$$

is uniformly convergent in $\mathfrak{B}$.

4°.

$$\lim_{b=\infty}\int_a^y h(b, y)dy = 0.$$

Then

$$J' = \frac{d}{dy}\int_a^\infty f(xy)dx = \int_a^\infty g(xy)dx. \qquad (1$$

2. As corollary we have:

Let $f(xy)$ *be regular in* $R = (a\infty\,\alpha\beta)$, *and continuous with respect to* y *for each* x *in* $\mathfrak{A}$. *Let* f_y' *be continuous in* R. *For any* b, *let*

$$\int_a^b f_y' dx = \int_a^b g(xy)dx + h(b, y),$$

where g *is regular in* R, *and*

$$\int_a^\infty g(xy)dx$$

is uniformly convergent in $\mathfrak{B}$; *also*

$$\lim_{b=\infty} \int_a^y h(b, y)dy = 0.$$

Then
$$\frac{d}{dy}\int_a^\infty f(xy)dx = \int_a^\infty g(xy)dx.$$

For, condition 1° of 683 is obviously satisfied, while condition 2° is fulfilled by 672, 2. Hence

$$J' = \frac{d}{dy}\lim_{b=\infty}\int_a^y dy\int_a^b f_y' dx.$$

But
$$\int_a^y dy \int_a^b f_y' dx = \int_a^y dy \int_a^b g dx + \int_a^y h dy, \qquad \text{by } 2°.$$

Hence by 4°,
$$J' = \frac{d}{dy}\lim \int_a^y dy \int_a^b g dx$$

$$= \frac{d}{dy}\int_a^y dx \int_a^\infty g dx, \qquad \text{by } 3°,$$

$$= \int_a^\infty g dx, \qquad \text{which is 1).}$$

EXAMPLES

685. 1. Let
$$J = \int_0^\infty \frac{\sin xy}{xe^x}dx. \tag{1}$$

We show that
$$\frac{dJ}{dy} = \int_0^\infty \frac{\cos xy}{e^x}dx, \qquad y \text{ arbitrary}, \tag{2}$$

using 683, 2. For, in the first place, the integrand $f(xy)$ is continuous in $R = (0\infty\,\alpha\beta)$, if we set

$$f(0, y) = y.$$

Obviously J is convergent in $\mathfrak{B}$, by 635, 3.

Secondly,

$$f'_y = \frac{\cos xy}{e^x}$$

is continuous in R; and

$$\int_a^\infty \frac{\cos xy}{e^x} dx$$

is uniformly convergent in $\mathfrak{B}$, by 660, 2. Thus 683, 2 gives **2)**.

By means of 2) we can evaluate 1).

For, obviously,

$$\int_0^\infty \frac{\cos xy}{e^x} dx = \frac{1}{1+y^2}.$$

Hence integrating 2), we get

$$J = \int \frac{dy}{1+y^2} = \text{arc tg}\, y + C.$$

Since $J = 0$, for $y = 0$, we have $C = 0$. Hence

$$\int_0^\infty \frac{\sin xy}{xe^x} dx = \text{arc tg}\, y. \tag{3}$$

2. From this integral we can also show that

$$\int_0^\infty \frac{\sin xy}{x} dx = \frac{\pi}{2}, \qquad y > 0, \tag{4}$$

a result obtained in 667, by the aid of 675. For, set

$$x = \frac{u}{y}, \qquad y > 0,$$

in 3), we get

$$\int_0^\infty \frac{\sin u}{u} \cdot e^{-\frac{u}{y}} du = \text{arc tg}\, y. \tag{5}$$

We now apply 666, 1, letting $y \doteq \infty$.

This is permissible, since

$$\frac{\sin u}{u} e^{-\frac{u}{y}} \doteq \frac{\sin u}{u}, \qquad \text{uniformly}$$

in $(0, \infty)$ except for $u = 0$. The integrand $f(u, y)$ is continuous in $R = (0\infty\alpha\infty)$, if we set

$$f(0, y) = 1.$$

The only singular line is therefore $u = 0$.

Obviously the singular integral for this line, as well as for the line $u = \infty$, is uniformly evanescent, by 615 and 659.

Hence passing to the limit, $y = \infty$ in 5), we get

$$\int_0^\infty \frac{\sin u}{u} du = \frac{\pi}{2}.$$

If we set $u = xy$, $y > 0$, we get 4).

686. Let

$$J = \int_0^\infty \frac{1 - \cos xy}{xe^x}\, dx. \tag{1}$$

Applying 683, 2, we get

$$\frac{dJ}{dy} = \int_0^\infty \frac{\sin xy}{e^x}\, dx = \frac{y}{1+y^2}, \qquad y \text{ arbitrary.} \tag{2}$$

In fact, the integrand $f(x, y)$ is continuous in $R = (0\infty\, \alpha\beta)$, if we set

$$f(0, y) = 0,$$

while J is convergent, by 635, 3.

Moreover

$$f_y' = \frac{\sin xy}{e^x}$$

is continuous in R, and

$$\int_0^\infty \frac{\sin xy}{e^x}\, dx$$

is uniformly convergent in $\mathfrak{B}$, by 660, 2. This establishes 2).

As in 685, we can use 2) to evaluate 1).

For, integrating 2), we get

$$J = \int \frac{y\, dy}{1+y^2} = \frac{1}{2} \log (1 + y^2) + C.$$

Here $C = 0$, since $J = 0$ for $y = 0$, by 1).

Thus

$$\int_0^\infty \frac{1 - \cos xy}{xe^x}\, dx = \frac{1}{2} \log (1 + y^2).$$

687. Let us evaluate *Fourier's Integral*

$$J = \int_0^\infty \frac{\cos 2\, xy}{e^{x^2}}\, dx. \tag{1}$$

Using 683, 2, we get

$$\frac{dJ}{dy} = -2 \int_0^\infty \frac{x \sin 2\, xy}{e^{x^2}}\, dx = K. \tag{2}$$

For, the integral 1) is convergent by 635, 2; while the integral 2) is uniformly convergent in any $(\alpha\beta)$, by 660, 2.

In 2), let us integrate by parts, setting

$$u = \sin 2\, xy, \qquad dv = -2\, xe^{-x^2} dx.$$

Then

$$K = uv \Big]_0^\infty - \int_0^\infty v\, du$$

$$= -2\, y \int_0^\infty e^{-x^2} \cos 2\, xy\, dx = -2\, yJ.$$

This in 2) gives, since $J \neq 0$,

$$\frac{dJ}{J} = -2\, y\, dy.$$

Hence

$$\log J = -y^2 + C. \tag{3}$$

To determine C, take $y = 0$. Then

$$C = \log \cdot \sqrt{\pi}/2, \tag{4}$$

by 682, 4).

Hence 1), 3), 4) give

$$\int_0^\infty \frac{\cos 2xy}{e^{x^2}}\,dx = \frac{\sqrt{\pi}}{2}\,e^{-y^2}.$$

688. In 681, 2) we found

$$\int_0^\infty \frac{\cos xy}{\mu^2 + x^2}\,dx = \frac{\pi}{2\,\mu e^{\mu y}}, \qquad \mu > 0, \qquad y \geqq \alpha > 0. \tag{1}$$

We can differentiate under the integral sign, by 683, 2. For, denoting the integrand by $f(xy)$, we have

$$f_y'(xy) = -\frac{x \sin xy}{\mu^2 + x^2},$$

which is continuous in $R = (0 \infty \alpha\beta)$.

Also

$$\int_0^\infty f_y' dx = -\int_0^\infty \frac{\sin xy}{x} \cdot \frac{dx}{1 + \dfrac{\mu^2}{x^2}}$$

is uniformly convergent in $\mathfrak{B}$, by 661, 2.

Hence, differentiating 1), we get

$$\int_0^\infty \frac{x \sin xy}{\mu^2 + x^2}\,dx = \frac{\pi}{2}\,e^{-\mu y}. \qquad \mu,\ y > 0.$$

689. In 682, 4), let us replace x by $xy^{\frac{1}{2}}$, $y > 0$. We get

$$\int_0^\infty e^{-yx^2} dx = \frac{\sqrt{\pi}}{2}\,y^{-\frac{1}{2}}, \qquad y \geq \alpha > 0. \tag{1}$$

We can differentiate under the integral sign, by 683, 2, getting

$$\int_0^\infty x^2 e^{-yx^2} dx = \frac{\sqrt{\pi}}{4}\,y^{-\frac{3}{2}}. \tag{2}$$

In fact, the integral on the left of 2) is uniformly convergent in $\mathfrak{B} = (\alpha, \beta)$, since

$$\frac{x^2}{e^{yx^2}} \leq \frac{x^2}{e^{\alpha x^2}}, \qquad \text{in } \mathfrak{B}.$$

We may obviously differentiate 1) n times, which gives

$$\int_0^\infty x^{2n} e^{-yx^2} dx = \frac{\sqrt{\pi}}{2} \cdot \frac{1}{2} \cdot \frac{3}{2} \cdots \frac{2n-1}{2}\,y^{-\frac{2n+1}{2}}, \qquad y > 0. \tag{3}$$

690. *Fresnel's Integrals.*

Let us start with the relation 689, 1),

$$\int_0^\infty e^{-yx^2}dx = \frac{\sqrt{\pi}}{2}y^{-\frac{1}{2}}, \qquad y>0. \tag{1}$$

Let

$$f(xy) = \frac{\sin y}{e^{yx^2}}, \qquad \text{for } x>0;$$

$$= 0, \qquad \text{for } x=0.$$

Then

$$J = \int_0^\infty dy \int_0^\infty f dx = \int_0^\infty dy \int_0^\infty \frac{\sin y}{e^{x^2 y}}dx = \frac{\sqrt{\pi}}{2}\int_0^\infty \frac{\sin y}{\sqrt{y}}dy, \tag{2}$$

since the integral on the right is convergent, by 646. We can invert the order of integration here, by 680, 1. For, $f(xy)$ is continuous in $R=(0\infty 0\infty)$, except on the line $x=0$. It has, moreover, no point of infinite discontinuity in R. The integral

$$\int_0^\infty f dx = \int_0^\infty \frac{\sin y}{e^{x^2 y}}dx$$

is uniformly convergent in any $(0, \beta)$ except at $y=0$. The integral

$$\int_0^\infty f dy = \int_0^\infty \frac{\sin y}{e^{x^2 y}}dy, \qquad x>0.$$

is uniformly convergent in any $(0, b)$, except at $x=0$. Finally,

$$Y = \int_0^\infty dx \int_0^y f dy$$

is uniformly convergent in $\mathfrak{B}$. For

$$\int_0^y f dy = \left[-\frac{x^2 \sin y + \cos y}{(1+x^4)e^{x^2 y}}\right]_0^y. \tag{3}$$

Hence

$$Y = \int_0^\infty \frac{dx}{1+x^4} - \int_0^\infty \frac{x^2\sin y + \cos y}{(1+x^4)e^{x^2 y}}dx = Y_1 - Y_2.$$

Here Y_1 is uniformly convergent in $\mathfrak{B}$, since it is independent of y. Likewise Y_2 is uniformly convergent, since its integrand is numerically

$$\leq \frac{1+x^2}{1+x^4}.$$

Thus all the conditions of 680, 1 being fulfilled, we can invert the order of integration in 2), which gives

$$J = \int_0^\infty dx \int_0^\infty f dy$$

$$= \int_0^\infty \frac{dx}{1+x^4}, \tag{4}$$

as is seen from 3), on passing to the limit $y=\infty$. But

$$\int_0^\infty \frac{dx}{1+x^4} = \frac{\pi}{4}\frac{1}{\sin \pi/4} = \frac{\pi}{2\sqrt{2}}.$$

This by 2), 4), gives

$$\int_0^\infty \frac{\sin y}{\sqrt{y}}\,dy = \frac{\sqrt{\pi}}{\sqrt{2}}. \tag{5}$$

If instead of multiplying 1) by $\sin y$, we had multiplied by $\cos y$, we would have got by the same reasoning

$$\int_0^\infty \frac{\cos y}{\sqrt{y}}\,dy = \frac{\sqrt{\pi}}{\sqrt{2}}. \tag{6}$$

The integrals 5), 6) are known as *Fresnel's integrals*. They occur in the Theory of Light.

If we set $y = x^2$, these integrals give

$$\int_0^\infty \sin x^2 dx = \int_0^\infty \cos x^2 dx = \tfrac{1}{2}\sqrt{\pi/2}.$$

691. 1. Let us show that Stoke's Integral

$$S = \int_0^\infty \cos(x^3 - xy)dx \tag{1}$$

satisfies the relation

$$\frac{d^2S}{dy^2} + \frac{1}{3}yS = 0. \tag{2}$$

This fact will enable us to compute S by means of an infinite series.

We have in the first place,

$$\frac{dS}{dy} = \int_0^\infty x\sin(x^3 - xy)dx \tag{3}$$

by 683, 2, since the integral 3) is uniformly convergent in any $\mathfrak{B} = (\alpha, \beta)$.

In fact, using the transformation of the variable employed in 657

$$u = x(x^2 - y), \tag{4}$$

we have

$$\int_b^\infty x\sin(x^3 - xy)dx = \int_c^\infty \frac{x\sin u\,du}{3x^2 - y},$$

where b, c are corresponding values in 4).

But

$$\frac{x\sin u}{3x^2 - y} = \frac{\sin u}{u}\cdot\frac{xu}{3x^2 - y} = \phi(u)g(u, y).$$

We can now apply 661, 1, replacing x in that theorem by u. Thus there exists a c_0 such that

$$\left|\int_c^\infty \frac{x\sin u\,du}{3x^2 - y}\right| < \epsilon, \qquad c \gtreqless c_0.$$

But then the relation 4) shows that there exists a B such that

$$\left|\int_b^\infty x\sin(x^3 - xy)dx\right| < \epsilon,$$

for any $b \gtrless B$, and for any y in $\mathfrak{B}$. Hence the integral 3) is uniformly convergent.

To find the second derivative of S, we cannot apply 683 to the integral 3). For

$$\int_0^\infty x^2 \cos(x^3 - xy)dx$$

is not even convergent, as we saw 657.

We may, however, apply 684, 2. In fact,

$$\int_0^b x^2 \cos(x^3 - xy)dx = \int_0^b \frac{3x^2 - y}{3} \cos(x^3 - xy)dx + \frac{y}{3}\int_0^b \cos(x^3 - xy)dx$$

$$= \tfrac{1}{3}\left[\sin(x^3 - xy)\right]_0^b + Y$$

$$= \tfrac{1}{3}\sin(b^3 - by) + Y.$$

But

$$\int_0^\infty \cos(x^3 - xy)dx$$

is uniformly convergent, as we saw 663, 7.

On the other hand,

$$\int_a^y \sin(b^3 - by)dy = \frac{\cos(b^3 - by) - \cos(b^3 - ba)}{b},$$

which $\doteq 0$ as $b \doteq \infty$. Thus 684, 2 gives

$$\frac{d}{dy}\int_0^\infty x \sin(x^3 - xy)dx = \frac{-y}{3}\int_0^\infty \cos(x^3 - xy)dx. \qquad (5$$

From 1), 3), 5) we have 2).

2. Before leaving this subject, let us show the uniform convergence of the integral 3), by another method.

From the identity

$$x = \frac{3x^2 - y}{3x} + \frac{y}{3}\,\frac{3x^2 - y}{3x^3} + \frac{y^2}{9x^3},$$

we have

$$\int_b^\infty x \sin(x^3 - xy)dx = \int_b^\infty \frac{3x^2 - y}{3x}\sin(x^3 - xy)dx + \frac{y}{3}\int_b^\infty \frac{3x^2 - y}{3x^3}\sin(x^3 - xy)dx$$

$$+ \frac{y^2}{9}\int_b^\infty \frac{\sin(x^3 - xy)}{x^3}dx = T_1 + T_2 + T_3.$$

Obviously T_3 is uniformly convergent by 660, 2.

That T_1 is uniformly convergent, was shown in 663, 6. That T_2 is uniformly convergent, follows from 661, 2; since T_1 is uniformly convergent.

Elementary Properties of $B(u, v)$, $\Gamma(u)$

692. 1. In 641 we saw

$$B(u, v) = \int_0^\infty \frac{x^{u-1}dx}{(1 + x)^{u+v}} \qquad (1$$

is a one-valued function whose domain of definition is the first quadrant in the u-, v-plane, points on the u-, v-axes excepted.

In 642 we saw
$$\Gamma(u) = \int_0^\infty e^{-x} x^{u-1} dx \tag{2}$$

is a one-valued function whose domain of definition is the positive half of the u-axis, the origin excepted. We wish to deduce here a few of the elementary properties of these functions.

2. By a change of variable, the integrals 1), 2) take on various forms. Thus in 1) set
$$x = \frac{y}{1-y}.$$

We get
$$B(u, v) = \int_0^1 y^{u-1}(1-y)^{v-1} dy. \tag{3}$$

If we set here
$$y = 1 - z,$$

we get
$$B(u, v) = \int_0^1 z^{v-1}(1-z)^{u-1} dz. \tag{4}$$

In 3) let us set $y = \sin^2\theta$; we get
$$B(u, v) = 2\int_0^{\frac{\pi}{2}} \sin^{2u-1}\theta \cos^{2v-1}\theta \, d\theta. \tag{5}$$

If we set
$$x = \log 1/y$$

in 2), we get
$$\Gamma(u) = \int_0^1 \log\left(\frac{1}{y}\right)^{u-1} dy. \tag{6}$$

3. We establish now a few relations for the B functions. In the first place the comparison of 3), 4) gives
$$B(u, v) = B(v, u), \tag{7}$$

which shows that B is symmetric in both its arguments.

As addition formulæ we have the three following 8), 9), 10),
$$B(u+1, v) + B(u, v+1) = B(u, v) \tag{8}$$

For,
$$B(u, v) = \int_0^1 x^{u-1}(1-x)^{v-1}(1-x+x)dx$$
$$= \int_0^1 x^u(1-x)^{v-1}dx + \int_0^1 x^{u-1}(1-x)^v dx,$$

which is 8).
$$vB(u+1, v) = uB(u, v+1). \tag{9}$$

For,
$$B(u+1, v)=\int_0^1 x^u(1-x)^{v-1}dx;$$

integrating by parts,
$$=\left[-\frac{x^u(1-x)^v}{v}\right]_0^1+\frac{u}{v}\int_0^1 x^{u-1}(1-x)^v dx$$

$$=\frac{u}{v}B(u, v+1),$$

which is 9).

From 8), 9) we have

$$B(u, v)=\frac{u+v}{v}B(u, v+1)=\frac{u+v}{u}B(u+1, v). \qquad (10$$

We can show now that $B(u, n)=B(n, u)$ is a rational function of u, viz.:

$$B(u, 1)=1/u. \qquad (11$$

$$B(u, n)=\frac{1}{u}\cdot\frac{1}{u+1}\cdot\frac{2}{u+2}\cdot\cdots\frac{n-1}{u+n-1}. \qquad (12$$

For,
$$B(u, 1)=\int_0^1 x^{u-1}dx=\left[\frac{x^u}{u}\right]_0^1=\frac{1}{u},$$

which proves 11). From this we get 12), using 10).

4. We establish now a few relations for the Γ function.

$$\Gamma(u+1)=u\Gamma(u). \qquad (13$$

For, integrating by parts,

$$\Gamma(u+1)=\int_0^\infty x^u e^{-x}dx=\left[-e^{-x}x^u\right]_0^\infty+u\int_0^\infty e^{-x}x^{u-1}dx$$

$$=u\int_0^\infty e^{-x}x^{u-1}dx.$$

We observe next that
$$\Gamma(1)=1. \qquad (14$$

For,
$$\Gamma(1)=\int_0^\infty e^{-x}dx=\left[-e^{-x}\right]_0^\infty=1.$$

From 13), 14), we get

$$\Gamma(u+n)=u(u+1)\cdots(u+n-1)\Gamma(u); \qquad (15$$

and this gives
$$\Gamma(n)=1\cdot 2\cdot 3\cdots n-1=n-1!, \qquad (16$$

on replacing n by $n-1$ and u by 1.

A formula occasionally useful is

$$\frac{1}{a^u}=\frac{1}{\Gamma(u)}\int_0^\infty e^{-ax}x^{u-1}dx. \tag{17}$$

It is obtained from 2) by replacing x, by ax.

5. The Γ function is continuous for any $u>0$. This follows from 669, 1 and 663, Ex. 1.

The derivative is given by

$$\Gamma'(u)=\int_0^\infty e^{-x}x^{u-1}\log x\,dx, \qquad u>0. \tag{18}$$

This follows from 683, 2. Similarly

$$\Gamma''(u)=\int_0^\infty e^{-x}x^{u-1}\log^2 x\,dx, \qquad u>0. \tag{19}$$

We can now get a good idea of the graph of $\Gamma(u)$. In fact, the expression 2) shows that $\Gamma(u)>0$ for all $u>0$.

From

$$\Gamma(u)=\int_0^1+\int_1^\infty$$

we see that

$$R\lim_{u=0}\Gamma(u)=+\infty.$$

From 13) we see that

$$\lim_{u=+\infty}\Gamma(u)=+\infty.$$

From 19) we see that $\Gamma''(u)>0$, and hence the graph of $\Gamma(u)$ is concave.

Since $\Gamma(1)=\Gamma(2)$, the curve has a minimum between 1 and 2. Its value is

$$1.46163\cdots$$

6. We establish now the important relation connecting the B and Γ functions,

$$B(u, v)=\frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)}. \tag{20}$$

From 17) we have

$$\frac{1}{(1+y)^{u+v}}=\frac{1}{\Gamma(u+v)}\int_0^\infty e^{-(1+y)x}x^{u+v-1}dx.$$

Hence by 1)

$$B(u, v) = \int_0^\infty \frac{y^{u-1}dy}{(1+y)^{u+v}} = \frac{1}{\Gamma(u+v)} \int_0^\infty dy \int_0^\infty x^{u+v-1} y^{u-1} e^{-(1+y)x} dx. \qquad (21$$

We may invert the order of integration, by 680, 3.

For, in the first place

$$f(xy) = \frac{x^{u+v-1} y^{u-1}}{e^{(1+y)x}}$$

is continuous in $R = (0\infty 0\infty)$, except on the lines $x = 0$, $y = 0$.

Secondly,

$$\int_0^\infty f dx$$

is uniformly convergent in any (α, β), $\alpha > 0$, by 663, Ex. 1.

Thirdly,

$$\int_0^\infty f dy$$

is uniformly convergent in any (a, b), $a > 0$.

Finally,

$$L = \int_0^\infty dx \int_0^\infty f dy$$

exists. For in

$$Y = \int_0^\infty \frac{y^{u-1}dy}{e^{(1+y)x}},$$

set $xy = t$, $x > 0$. Then

$$Y = e^{-x} x^{-u} \int_0^\infty e^{-t} t^{u-1} dt$$

$$= e^{-x} x^{-u} \Gamma(u).$$

Hence for $a > 0$,

$$L_a = \int_a^\infty dx \int_0^\infty f dy = \Gamma(u) \int_a^\infty e^{-x} x^{v-1} dx.$$

But

$$\lim_{a=0} \int_a^\infty e^{-x} x^{v-1} dx = \Gamma(v).$$

Hence

$$L = \lim_{a=0} L_a = \Gamma(u)\Gamma(v). \qquad (22$$

Thus all the conditions of 680, 3 being fulfilled, we have $L = K$. From 21), 22), we have 18).

CHAPTER XVI

MULTIPLE PROPER INTEGRALS

Notation

693. 1. In Chapters XII and XIII the theory of proper integrals of functions of one variable was developed. We now take up the corresponding theory with reference to functions of several variables.

2. We begin by explaining a notation which we shall systematically employ in the following, and which is similar to that used in the earlier chapters.

Let $\mathfrak{A}$ be a limited point aggregate in an m-way space $\Re_m$. Let $f(x_1, \cdots x_m)$, or as we shall often write it, $f(x)$, be a limited function defined over $\mathfrak{A}$. Let us effect a rectangular division D of space of norm d. To simplify matters, we shall suppose d is not taken larger than some arbitrarily large but fixed number. Those cells which contain points of $\mathfrak{A}$, as well as their volumes, will be denoted by $d_1, d_2, \cdots$, or by a similar notation. Let M_ι, m_ι, be the maximum and minimum of $f(x)$ in d_ι. We shall set

$$\overline{S}_D = \Sigma M_\iota d_\iota, \qquad \underline{S}_D = \Sigma m_\iota d_\iota. \tag*{(1}$$

It sometimes happens that we are considering points of two or more aggregates $\mathfrak{A}, \mathfrak{B}, \cdots$ Then we shall write

$$\overline{S}_{\mathfrak{A}_D}, \ \overline{S}_{\mathfrak{B}_D}, \cdots$$

where the subscript indicates that the sums 1) are taken over the aggregates $\mathfrak{A}, \mathfrak{B}, \cdots$ respectively.

3. We shall denote the maximum and minimum of f in $\mathfrak{A}$ by M and m respectively. The greater of $|M|$ and $|m|$ we shall denote by F, so that

$$|f(x_1, \cdots x_m)| \leq F, \qquad \text{in } \mathfrak{A}.$$

4. The *oscillation* of $f(x_1, \cdots x_m)$ in the cell d_ι is

$$\omega_\iota = M_\iota - m_\iota.$$

The sum

$$\Omega_D f = \Sigma \omega_\iota d_\iota = \overline{S}_D - \underline{S}_D$$

is the *oscillatory sum of f for the division D.*

Upper and Lower Integrals

694. *The sums $\underline{S}_D$, $\overline{S}_D$ form a limited aggregate, D representing any division of norm $\leq d_0$; moreover*

$$\underline{S}_D \leq \overline{S}_D.$$

For

$$m \leq m_\iota \leq M_\iota \leq M.$$

Hence

$$\Sigma m d_\iota \leq \Sigma m_\iota d_\iota \leq \Sigma M_\iota d_\iota \leq \Sigma M d_\iota,$$

or

$$m \Sigma d_\iota \leq \underline{S}_D \leq \overline{S}_D \leq M \Sigma d_\iota. \qquad (1$$

Since $\mathfrak{A}$ is limited, the cells d_ι are all contained in some cube. Hence Σd_ι is less than some fixed number, and the theorem follows at once from 1).

695. 1. *Let $f(x_1, \cdots x_m) \gtreqless 0$ in $\mathfrak{A}$. Let D and Δ be any two rectangular divisions of space. Let E be the division of space formed by superimposing the division Δ on D, or what is the same, the division D on Δ. Then*

$$\overline{S}_E \leq \overline{S}_D, \qquad \overline{S}_E \leq \overline{S}_\Delta.$$

For, let d_ι be one of the cells of D which is subdivided, on superimposing Δ.

Let

$$d_{\iota 1}, \ d_{\iota 2}, \cdots$$

denote the cells of E in d_ι containing points of $\mathfrak{A}$. Then, to the term $M_\iota d_\iota$ in $\overline{S}_D$, corresponds the term

$$\sum_\kappa M_{\iota\kappa} d_{\iota\kappa}$$

in $\overline{S}_E$. But

$$M_{\iota\kappa} \leq M_\iota, \qquad \sum_\kappa d_{\iota\kappa} \leq d_\iota.$$

Hence

$$\sum_\kappa M_{\iota\kappa} d_{\iota\kappa} \leq \Sigma M_\iota d_{\iota\kappa} \leq M_\iota d_\iota.$$

2. Similar reasoning shows that:

If $f(x_1, \cdots x_m) \overset{=}{<} 0$ *in* $\mathfrak{A}$,

$$\underline{S}_D \overset{=}{<} \underline{S}_E, \qquad \underline{S}_\Delta \overset{=}{<} \underline{S}_E.$$

696. 1. *Let* $f(x_1, \cdots x_m) \overset{=}{>} 0$ *be limited in the limited aggregate* $\mathfrak{A}$. *Let*

$$\overline{S} = \text{Min } \overline{S}_D,$$

with respect to all rectangular divisions D. *Then*

$$\lim_{d=0} \overline{S}_D = \overline{S}. \tag{1}$$

Let us employ the graphical representation of 231. The points of $\mathfrak{A}$ lie in a certain cube $\mathfrak{C}$ of edge C. The representation of $\mathfrak{C}$ is formed of m segments $\mathfrak{C}_1, \cdots \mathfrak{C}_m$ on the $x_1, \cdots x_m$ axes. We shall suppose $\mathfrak{C}$ taken so large that no coördinate of any point of $\mathfrak{A}$ is at a distance $\leq 2d_0$ from the ends of these segments. This insures that the cells d_ι of any D of norm $\leq d_0$ lie within $\mathfrak{C}$, and therefore that $\Sigma d_\iota < C^m$.

Since $\overline{S}$ is the minimum of all $\overline{S}_D$, there exists for each $\epsilon > 0$ a division Δ, such that

$$\overline{S} \leq \overline{S}_\Delta > \overline{S} + \epsilon/2. \tag{2}$$

Let D be an arbitrary division. Let us superimpose Δ on D, forming a division E.

The division E is formed by interpolating certain points, let us say at most μ points in each of the segments $\mathfrak{C}_1, \cdots \mathfrak{C}_m$. The interpolation of one of these points may be interpreted as passing a plane parallel to one of the sides of $\mathfrak{C}$. Its effect is to subdivide certain of the cells of $\mathfrak{C}$. The volume of the cells so affected is

$$\leq d\, C^{m-1}.$$

Hence the superimposition of Δ on D, being equivalent to passing at most $m\mu$ planes parallel to the sides of $\mathfrak{C}$, affects cells of $\mathfrak{C}$ belonging to the original division D, whose volume

$$V < m\mu d\, C^{m-1}. \tag{3}$$

Let Δ subdivide d_ι, a cell of D containing points of $\mathfrak{A}$, into the cells

$$d_{\iota 1},\ d_{\iota 2} \cdots$$

containing points of $\mathfrak{A}$, and into the cells

$$\delta_{\iota 1},\ \delta_{\iota 2} \cdots$$

containing no point of $\mathfrak{A}$.

Then

$$\overline{S}_D = \sum_\iota M_\iota d_\iota + R\,;\quad \overline{S}_E = \sum_{\iota\kappa} M_{\iota\kappa} d_{\iota\kappa} + R,$$

where R denotes the sum of those terms common to $\overline{S}_D$ and $\overline{S}_E$, corresponding to cells of D unaffected by the division Δ.

But

$$d_\iota = \sum_{\iota\kappa} d_{\iota\kappa} + \sum_{\iota\kappa} \delta_{\iota\kappa}.$$

Hence

$$\overline{S}_D = \sum_{\iota\kappa} M_\iota d_{\iota\kappa} + \sum_{\iota\kappa} M_\iota \delta_{\iota\kappa} + R.$$

Therefore

$$\begin{aligned} 0 \leq \overline{S}_D - \overline{S}_E &= \Sigma(M_\iota - M_{\iota\kappa}) d_{\iota\kappa} + \Sigma M_\iota \delta_{\iota\kappa} \\ &\leq F \Sigma d_{\iota\kappa} + F \Sigma \delta_{\iota\kappa} = FV \\ &< m\mu d F C^{m-1}, \quad \text{by 3)}. \end{aligned}$$

If we take

$$d' < \frac{\epsilon}{2\, m\mu F C^{m-1}},$$

we have

$$\overline{S}_D < \overline{S}_E + \epsilon/2, \qquad \text{for any } d \leq d'. \tag{4}$$

But regarding E as formed by superimposing D on Δ,

$$\overline{S}_E < \overline{S}_\Delta. \tag{5}$$

Hence 2), 4), 5) give

$$\overline{S} \leq \overline{S}_D < \overline{S}_\Delta + \epsilon/2 < \overline{S} + \epsilon\,;$$

or

$$0 \leq \overline{S}_D - \overline{S} < \epsilon,$$

which proves 1).

2. A similar line of reasoning shows:

Let $f(x_1 \cdots x_m) \gtreqless 0$ *be limited in the limited aggregate* $\mathfrak{A}$. *Let*

$$\underline{S} = \operatorname{Max} \underline{S}_D,$$

with respect to all rectangular divisions D. *Then*

$$\lim_{d=0} \underline{S}_D = \underline{S}.$$

697. *Let $f(x_1 \cdots x_m)$ be limited in the limited aggregate $\mathfrak{A}$. Then the limits*

$$\lim_{d=0} \underline{S}_D, \qquad \lim_{d=0} \overline{S}_D$$

exist, and are finite.

Let us take $c > 0$ so large that

$$g(x_1 \cdots x_m) = f(x_1 \cdots x_m) + c$$

is positive. Let M_ι, N_ι be respectively the maxima of f and g in the cell d_ι. Obviously,

$$N_\iota = M_\iota + c.$$

We have seen in 696, 1 that

$$\lim \Sigma N_\iota d_\iota, \qquad \lim \Sigma c d_\iota$$

exist. Hence

$$\lim_{d=0} \overline{S}_D = \lim \Sigma M_\iota d_\iota = \lim \Sigma (N_\iota - c) d_\iota$$
$$= \lim \Sigma N_\iota d_\iota - \lim \Sigma c d_\iota$$

exists, and is finite.

To show that

$$\lim_{d=0} \underline{S}_D$$

exists and is finite, we introduce the auxiliary function

$$h(x_1 \cdots x_m) = f(x_1 \cdots x_m) - c;$$

and determine $c > 0$ so large that h is always negative in $\mathfrak{A}$.

698. The limits $\underline{S}$, $\overline{S}$, whose existence was established in 697, are called the *lower and upper integrals of $f(x_1 \cdots x_m)$ over the field* $\mathfrak{A}$. They are denoted respectively by

$$\underline{\int}_{\mathfrak{A}} f(x_1 \cdots x_m) d\mathfrak{A} = \underline{\int}_{\mathfrak{A}} f(x_1 \cdots x_m) dx_1 \cdots dx_m;$$
$$\overline{\int}_{\mathfrak{A}} f(x_1 \cdots x_m) d\mathfrak{A} = \overline{\int}_{\mathfrak{A}} f(x_1 \cdots x_m) dx_1 \cdots dx_m. \qquad (1$$

When the lower and upper integrals 1) are equal, we denote their common value by

$$\int_{\mathfrak{A}} f(x_1 \cdots x_m) d\mathfrak{A} = \int_{\mathfrak{A}} f(x_1 \cdots x_m) dx_1 \cdots dx_m; \qquad (2$$

it is called the *integral of* f *over the field* $\mathfrak{A}$. In this case $f(x_1 \cdots x_m)$ *is said to be integrable in* $\mathfrak{A}$. We also say the *integral* 2) *exists.*

The integrals 1), 2) are called *m-tuple* or *multiple integrals.*

699. 1. *Let* $f(x_1 \cdots x_m)$ *be limited and integrable in the limited field* $\mathfrak{A}$. *Let* D *be any rectangular division of norm* d, *and* ξ_ι *any point of* $\mathfrak{A}$ *in the cell* d_ι. *Then*

$$\lim_{d=0} \Sigma f(\xi_\iota)d_\iota = \int_{\mathfrak{A}} f d\mathfrak{A}. \tag{1}$$

Conversely, if this limit exists, however the D'*s and* ξ'*s be chosen, the upper and lower integrals of* f *are equal, and* f *is integrable.*

For,

$$m_\iota \leq f(\xi_\iota) \leq M_\iota.$$

Hence

$$\Sigma m_\iota d_\iota \leq \Sigma f(\xi_\iota) d_\iota \leq \Sigma M_\iota d_\iota. \tag{2}$$

As

$$\lim \Sigma m_\iota d_\iota = \underline{\int}_{\mathfrak{A}},$$

$$\lim \Sigma M_\iota d_\iota = \overline{\int}_{\mathfrak{A}}$$

are equal, we get 1) on passing to the limit $d = 0$ in 2).

The reader will observe that this reasoning is precisely similar to the first half of the demonstration in 493. The second half of our theorem is proved by a reasoning exactly similar to the second half of the demonstration of 493. Instead of the interval $b - a$, we have here a cube of volume C^m.

2. The theorem 1 shows us that we may take

$$\lim_{d=0} \Sigma f(\xi_\kappa) d_\kappa$$

when it exists as a *second definition of the integral of* f *over* $\mathfrak{A}$.

700. 1. The theorems of 495, 496, 497, and 498 may now be extended without trouble to functions of several variables. For convenience of reference we restate them here for this general case.

2. *In order that the limited function* $f(x_1 \cdots x_m)$ *be integrable in the limited field* $\mathfrak{A}$, *it is necessary and sufficient that the oscillatory sum* $\Omega_D f \doteq 0$, *as the norms of the divisions* D *converge to* 0.

3. *If the limited function* $f(x_1 \cdots x_m)$ *is integrable over the limited field* $\mathfrak{A}$, *it is integrable over any partial field of* $\mathfrak{A}$.

4. *In order that the limited function* $f(x_1 \cdots x_m)$ *be integrable in the limited field* $\mathfrak{A}$, *it is necessary and sufficient that for each* $\epsilon > 0$, *there exists a division* D *for which the oscillatory sum*

$$\Omega_D f < \epsilon.$$

5. *In order that the limited function* $f(x_1 \cdots x_m)$ *be integrable in the limited field* $\mathfrak{A}$, *it is necessary and sufficient that, for each pair of positive numbers* ω, σ *there exists a division* D, *such that the sum of the cells of* D *in which the oscillation of* f *is* $> \omega$, *is* $< \sigma$.

EXAMPLES

701. 1. Let $\mathfrak{A}$ be the square (0, 1, 0, 1).

Let $f(x, y) = 0$, for x, or y *irrational;*

$= \frac{1}{n}$, *for* $x = \frac{m}{n}$; m, n *relative prime*, y rational.

Then f is integrable, by 700, 5. For, f is $\geq \frac{1}{q}$ only on the lines $x = 1, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \cdots$, the denominators of the fractions being $\leqq q$. On each of these lines the oscillation in any little interval is $\geqq \frac{1}{q}$. On all other lines the oscillation is $< \frac{1}{q}$. Obviously there exists for each σ a division for which the sum of the squares in which the oscillation is $\geqq \frac{1}{q}$ is $< \sigma$; and the integral is zero.

2. Let $\mathfrak{A}$ embrace the points x, y of the square (0101) for which x is rational.

Let $f(x, y) = \frac{1}{n}$, for $x = \frac{m}{n}$; m, n relative prime.

Then f is integrable in $\mathfrak{A}$, as the above example shows.

Content of Point Aggregates

702. 1. We extend now the notion of content, etc., considered in 514 seq., to limited aggregates in $\mathfrak{R}_m$. Let us effect a rectangular division of space of norm δ. Let

$$d_1, \ d_2, \ d_3, \ \cdots$$

be those cells containing at least one point of the limited aggregate $\mathfrak{A}$; while

$$d'_1,\ d'_2,\ d',\ \cdots$$

denote those cells, *all* of whose points lie in $\mathfrak{A}$.

Then the limits

$$\overline{\mathfrak{A}} = \lim_{\delta=0} \Sigma d_\kappa, \qquad \underline{\mathfrak{A}} = \lim_{\delta=0} \Sigma d'_\kappa \tag{1}$$

exist, and are finite.

For, let us introduce the auxiliary function $f(x_1 \cdots x_m)$, whose value shall be 0 in $\mathfrak{R}_m$, except at the points of $\mathfrak{A}$, where its value is 1. Then, using the notation and results of the previous articles, we have:

$$\overline{\mathfrak{A}}_D = \Sigma M_\kappa d_\kappa = \Sigma d_\kappa,$$

$$\underline{\mathfrak{A}}_D = \Sigma m_\kappa d_\kappa = \Sigma d'_\kappa.$$

But by 697,

$$\lim_{\delta=0} \overline{\mathfrak{A}}_D, \qquad \lim_{\delta=0} \underline{\mathfrak{A}}_D$$

exist, and are finite.

2. The numbers $\overline{\mathfrak{A}}$, $\underline{\mathfrak{A}}$ are called the *upper* and *lower content* of $\mathfrak{A}$ We have thus:

$$\overline{\mathfrak{A}} = \overline{\int} f d\mathfrak{A}, \qquad \underline{\mathfrak{A}} = \underline{\int} f d\mathfrak{A}.$$

When $\overline{\mathfrak{A}} = \underline{\mathfrak{A}}$, their common value is called the *content* of $\mathfrak{A}$. We denote it by

$$\text{Cont } \mathfrak{A},$$

or when no ambiguity arises, by $\mathfrak{A}$.

To be more explicit it is often convenient to set

$$\overline{\mathfrak{A}} = \overline{\text{Cont}}\ \mathfrak{A}, \qquad \underline{\mathfrak{A}} = \underline{\text{Cont}}\ \mathfrak{A}.$$

A limited aggregate having content is *measurable.*

Thus, when $\mathfrak{A}$ is measurable,

$$\text{Cont } \mathfrak{A} = \int_{\mathfrak{A}} f d\mathfrak{A}.$$

The content of a measurable aggregate in $\mathfrak{R}_2$ is called its *area;* in $\mathfrak{R}_3$ the content is called *volume.* We shall also use the term *volume* in this sense, when $n > 3$.

3. As immediate consequence of the reasoning of 1, we have: *Let $\mathfrak{B}$ be a partial aggregate of $\mathfrak{A}$. Then*

$$\overline{\mathfrak{B}} \leq \overline{\mathfrak{A}}; \qquad \underline{\mathfrak{B}} \leq \underline{\mathfrak{A}}.$$

703. By the aid of the auxiliary function employed in 702 we can state at once criteria in order that $\mathfrak{A}$ is measurable.

1. *For $\mathfrak{A}$ to be measurable, it is necessary and sufficient that the sum of the cells containing both points of $\mathfrak{A}$, and points not in $\mathfrak{A}$ converge to 0, as the norm of the division $\doteq 0$.*

This follows from 700, 2.

2. *In order that $\mathfrak{A}$ be measurable, it is necessary and sufficient that for each $\epsilon > 0$, there exists a division such that the sum of the cells embracing both points of $\mathfrak{A}$ and not of $\mathfrak{A}$ is $< \epsilon$.*

This follows from 700, 4.

Frontier Points

704. 1. *The frontier $\mathfrak{F}$ of any aggregate $\mathfrak{A}$ is complete.*

For, let p be a limiting point of $\mathfrak{F}$.

Then in any $D_\delta^*(p)$, there are points of $\mathfrak{F}$. If $\mathfrak{f}$ is such a point, there are points not belonging to $\mathfrak{A}$ in any $D_\eta^*(\mathfrak{f})$. We may take η so small that D_η lies in D_δ. Hence p is a frontier point of $\mathfrak{A}$.

2. Let $\mathfrak{A}$ and $\mathfrak{B}$ be two point aggregates. Let

$$D = \text{Dist}\ (x, y) = \sqrt{(x_1 - y_1)^2 + \cdots + (x_m - y_m)^2}$$

be the *distance* between a point x of $\mathfrak{A}$ and a point y of $\mathfrak{B}$. Let δ be the minimum of D, as x runs over $\mathfrak{A}$, and y runs over $\mathfrak{B}$. Then $\delta \gtreqless 0$, and is finite. We say δ is the *distance of $\mathfrak{A}$ from $\mathfrak{B}$*, and write

$$\delta = \text{Dist}\ (\mathfrak{A}, \mathfrak{B}).$$

In certain cases, $\mathfrak{A}$ may reduce to a single point a.

3. *If $\mathfrak{A}$, $\mathfrak{B}$ are limited and complete, there is a point a in $\mathfrak{A}$, and a point b in $\mathfrak{B}$, such that*

$$\text{Dist}\ (a, b) = \text{Dist}\ (\mathfrak{A}, \mathfrak{B}).$$

If Dist $(\mathfrak{A}, \mathfrak{B}) > 0$, the two points a, b are frontier points.

For, we may regard $x_1 \cdots x_m$, $y_1 \cdots y_m$ as the coördinates of a point z in a $2m$-way space $\mathfrak{R}_{2m}$. We form an aggregate $\mathfrak{C}$ whose points z are obtained by associating with each x of $\mathfrak{A}$, every y of $\mathfrak{B}$. Then the domain of definition of Dist (x, y) in 2, considered as a function of $2m$ variables, is precisely $\mathfrak{C}$. To represent $\mathfrak{C}$ we may employ $2m$ axes, as in 231. Obviously $\mathfrak{C}$ is limited and complete, since $\mathfrak{A}$ and $\mathfrak{B}$ are.

Then by 269, 2, there exists a point $(a_1 \cdots a_m, b_1 \cdot \cdot b_m)$ in $\mathfrak{C}$, at which D takes on its minimum value. Then

$$a = (a_1 \cdots a_m), \qquad b = (b_1 \cdots b_m)$$

are the points whose existence was to be proved.

The points a, b are *frontier points* of $\mathfrak{A}$ and $\mathfrak{B}$ respectively. For, if they were inner points, the distance between $D_\delta(a)$ and $D_\delta(b)$ equals

$$\text{Dist}\,(a, b) - 2\,\delta < \text{Dist}\,(a, b).$$

4. Let $\mathfrak{B}$ be a partial aggregate of $\mathfrak{A}$. If the distance between the frontiers of $\mathfrak{A}$ and $\mathfrak{B}$ is not 0, we say $\mathfrak{B}$ is an *inner partial aggregate of* $\mathfrak{A}$; also $\mathfrak{A}$ is an *outer aggregate of* $\mathfrak{B}$.

Discrete Aggregates

705. 1. *Definition.* An aggregate of content 0 is *discrete.*

Obviously, if

$$\overline{\text{Cont}}\ \mathfrak{A} = 0,$$

$\mathfrak{A}$ is discrete.

2. *Every limited point aggregate of the first species is discrete.*

Let $\mathfrak{A}$ embrace at first, only a finite number of points, say n points. Let us effect a cubical division of space of norm

$$\delta < \sqrt[m]{\frac{\epsilon}{n}},$$

such that the points of $\mathfrak{A}$ lie *within* their respective cells. Then the sum of the cells containing the points $\mathfrak{A}$ is

$$\Sigma v_\kappa \leq n\delta^m < \epsilon.$$

Thus $\mathfrak{A}$ is discrete, and the theorem is true for aggregates of order 0. Let us therefore assume the theorem is true for aggregates of order $n-1$ and show it is true for order n.

By 265, $\mathfrak{A}^{(n)}$ embraces only a finite number of points, say

$$a_1, a_2 \cdots a_s.$$

We can, as just seen, inclose these within cells of total volume $\leq \epsilon/2$. The points of $\mathfrak{A}$ not in these cells form an aggregate $\mathfrak{B}$ of order $n-1$. By hypothesis we can effect a division of space, such that the total volume of the cells containing both points of $\mathfrak{B}$ and not of $\mathfrak{B}$ is $< \epsilon/2$. Thus the division formed by superimposing these two divisions, is such that the volume of the cells containing both points of $\mathfrak{A}$ and not of $\mathfrak{A}$ is $< \epsilon$.

706. *Let $\mathfrak{A}$ be a limited aggregate whose frontier points $\mathfrak{F}$ form a discrete aggregate. Then $\mathfrak{A}$ is measurable.*

For, using the notation of 702, the volume of those cells of a division D, containing both points of $\mathfrak{A}$ and not of $\mathfrak{A}$, is

$$\overline{\mathfrak{A}}_D - \underline{\mathfrak{A}}_D \leq \overline{\mathfrak{F}}_D;$$

where $\overline{\mathfrak{F}}_D$ is the volume of those cells containing at least a point of $\mathfrak{F}$. But, as $\mathfrak{F}$ is discrete,

$$\mathfrak{F}_D \doteq 0.$$

Hence, by 703, 1, $\mathfrak{A}$ is measurable.

707. 1. Let $\mathfrak{R}_m$ be an m-way space. Let us give certain of the coördinates of $x = (x_1, \cdots x_m)$ fixed values. For example, let $x_{p+1} = a_{p+1}, \cdots x_m = a_m$. The aggregate of points $x = (x_1, \cdots x_p, a_{p+1}, \cdots a_m)$ may be regarded as constituting a *p-way space, $\mathfrak{R}_p$ lying in $\mathfrak{R}_m$*. The point x, when considered as belonging to $\mathfrak{R}_p$, may be denoted more shortly by $x = (x_1, \cdots x_p)$.

2. *Let $\mathfrak{A}$ be a limited aggregate lying in $\mathfrak{R}_p$. If we consider $\mathfrak{A}$ as lying in an m-way space $\mathfrak{R}_m$, $m > p$, it is discrete.*

For, let $\mathfrak{A}$ lie in a cube C, of volume C, in $\mathfrak{R}_p$, so large that none of the points of $\mathfrak{A}$ come indefinitely near the sides of C. Then the upper content of $\mathfrak{A}$, relative to $\mathfrak{R}_p$, is $< C$. We can effect a division D of $\mathfrak{R}_m$ of norm d such that the points of $\mathfrak{A}$ lie within the cells of D. Then the volume of all the cells containing points of $\mathfrak{A}$ is less than

$$Cd^{m-p},$$

which converges to 0, with d.

708. 1. Let $y=f(x_1, \cdots x_m)$ be defined over an aggregate $\mathfrak{A}$. Let $x=(x_1, \cdots x_m)$, $x'=(x_1+h_1, \cdots x_m+h_m)$ be two points of $\mathfrak{A}$. The increment that f receives when x passes to x' we have denoted by Δf. Let us set

$$\Delta x = \text{Dist}\,(x, x') = \sqrt{h_1^2 + \cdots + h_m^2},$$

and call

$$\frac{\Delta f}{\Delta x}$$

the *total difference quotient* of f. The point x' may or may not be restricted to remain near x; if so, it will be stated.

2. *Let the limited functions*

$$y_1 = f_1(x_1, \cdots x_m), \cdots y_n = f_n(x_1, \cdots x_m), \qquad n = m + p \geq m$$

have limited total difference quotients in the limited discrete aggregate $\mathfrak{A}$. *Then* $\mathfrak{B}$, *the y-image of* $\mathfrak{A}$, *is also discrete.*

For, let us effect a cubical division of the x-space of norm d.

Since the difference quotients are limited in $\mathfrak{A}$, there exists a fixed G, such that

$$|\Delta f_\iota| < dG, \qquad \iota = 1, 2, \cdots n,$$

as x ranges over any one of the cells d_ι of D. Hence each coördinate y_ι remains in an interval of length $< dG$ as x ranges over the points of $\mathfrak{A}$ in d_ι. Therefore $y=(y_1, \cdots y_n)$ remains within a cube of volume d^nG^n. Hence the points of $\mathfrak{B}$ have an upper content

$$< \sum_{\mathfrak{B}} d^n G^n = d^p G^n \sum_{\mathfrak{A}} d^m = d^p G^n \overline{\mathfrak{A}}_D.$$

As $\lim \overline{\mathfrak{A}}_D = 0$,

$$\text{Cont}\ \mathfrak{B} = 0.$$

3. As a corollary of 2 we have:

In the region R *let*

$$y_1 = f_1(x_1, \cdots x_m), \cdots y_n = f_n(x_1, \cdots x_m) \qquad n \gtreqless m$$

have limited first partial derivatives.

Let $\mathfrak{A}$ *be a limited inner discrete aggregate. Then* $\mathfrak{B}$, *the image of* $\mathfrak{A}$, *is discrete.*

4. *Let the limited functions*

$$y_1 = f_1(x_1, \cdots x_m), \cdots y_n = f_n(x_1, \cdots x_m) \qquad n = m + p > m$$

have limited total difference quotients in the limited aggregate $\mathfrak{A}$, *except at points of a discrete aggregate* Δ. *In the cells of any cubical division of norm* $d \leq d_0$, *let at least* m *of these difference quotients remain limited. Then the image* $\mathfrak{B}$ *of* $\mathfrak{A}$ *is discrete.*

For, consider one of the cells d_ι, containing a point of Δ. At least m of the coördinates of a point y remain in intervals of length

$$< Gd.$$

All we can say of the other coördinates of y is that they remain in intervals of length $2F$, where $|f_\iota| < F$, $\iota = 1, 2, \cdots n$. Thus the image of the points of $\mathfrak{A}$ in the cells d_ι has an upper content

$$< \sum_\iota (Gd)^m (2F)^p = G^m 2^p F^p \sum_\iota d^m = 2^p F^p G^m \bar{\Delta}_D < \epsilon/2,$$

if d_0 is taken small enough.

The content of the image of the other cells d_κ is

$$< \sum_\kappa d^n G^n \leq d^p G^n \bar{\mathfrak{A}}_D.$$

As $p \geqq 1$, we can take d_0 sufficiently small, so that the content of these cells is $< \epsilon/2$.

709. An important class of discrete aggregates is connected with functions having limited variation, which we now define. Cf. 509 seq.

Let $f(x_1 \cdots x_m)$ be limited in the limited aggregate $\mathfrak{A}$. Let D be a cubical division of space of norm $d \leq d_0$. Let the oscillation of f in the cell d_κ be ω_κ. If there exists a number $\tilde{\omega}$ such that

$$\Sigma \omega_\kappa d^{m-1} \leq \tilde{\omega}, \tag{1}$$

however D is chosen, we say that $f(x_1 \cdots x_m)$ *has limited variation in* $\mathfrak{A}$; otherwise it has *unlimited variation.*

From 1) we have

$$\Sigma \omega_\kappa \leq \frac{\tilde{\omega}}{d^{m-1}}. \tag{2}$$

710. *Let the limited functions*

$$y_1 = f_1(x_1 \cdots x_m) \cdots y_{n-1} = f_{n-1}(x_1 \cdots x_m) \qquad n = m + p > m$$

have limited total difference quotients in the limited aggregate $\mathfrak{A}$. *Let*

$$y_n = f_n(x_1 \cdots x_m)$$

have limited variation in $\mathfrak{A}$. *As* $x = (x_1 \cdots x_m)$ *ranges over* $\mathfrak{A}$, *let* $y = (y_1 \cdots y_n)$ *range over* $\mathfrak{B}$. *Then* $\mathfrak{B}$ *is discrete.*

For, let us effect a division of the x-space of norm d. Then $y_1, \cdots y_{n-1}$ remain in intervals of length $< dG$ as x ranges over the points of $\mathfrak{A}$ in one of the cells d_κ. Thus if ω_κ is the oscillation of f_n in d_κ, the point y remains in a cube of volume

$$< d^{n-1} G^{n-1} \omega_\kappa,$$

when x ranges in d_κ. Thus the upper content of $\mathfrak{B}$ is

$$< d^{n-1} G^{n-1} \Sigma \omega_\kappa$$

$$< d^p G^{n-1} \omega, \quad \text{by 709, 2).}$$

As this converges to 0 as $d \doteq 0$, $\mathfrak{B}$ is discrete.

Properties of Content

711. **1.** Let $\mathfrak{A}$ be a limited aggregate. With the points of $\mathfrak{A}$ let us form the partial aggregates $\mathfrak{A}_1, \cdots \mathfrak{A}_s$, such that the aggregate of the common points, or of the common frontier points, of any two of these aggregates is discrete. We shall say that *we have divided* $\mathfrak{A}$ *into the unmixed aggregates* $\mathfrak{A}_1, \cdots \mathfrak{A}_s$. Also, $\mathfrak{A}$ is *the union of* $\mathfrak{A}_1, \cdots \mathfrak{A}_s$.

2. *Let the limited aggregate* $\mathfrak{A}$ *be divided in the unmixed aggregates* $\mathfrak{A}_1, \mathfrak{A}_2, \cdots \mathfrak{A}_s$. *Then*

$$\overline{\mathfrak{A}} = \overline{\mathfrak{A}}_1 + \cdots + \overline{\mathfrak{A}}_s; \quad \underline{\mathfrak{A}} = \underline{\mathfrak{A}}_1 + \cdots + \underline{\mathfrak{A}}_s.$$

For, let D be a rectangular division of norm δ. Let $\overline{\mathfrak{F}}_D$ be the volume of all those cells of D which contain points of more than one of the aggregates $\mathfrak{A}_1, \cdots \mathfrak{A}_s$. Let $\overline{\mathfrak{A}}_{\iota, D}$ be the volume of those cells containing points of $\mathfrak{A}_\iota$, $\iota = 1, 2, \cdots s$. Then

$$\overline{\mathfrak{A}}_D \overset{=}{<} \overline{\mathfrak{A}}_{1D} + \cdots + \overline{\mathfrak{A}}_{s, D} \leq \overline{\mathfrak{A}}_D + s\overline{\mathfrak{F}}_D. \qquad (1$$

Now, by hypothesis,
$$\lim_{\delta=0} \overline{\mathfrak{F}}_D = 0.$$

Hence passing to the limit in 1), we get
$$\overline{\mathfrak{A}} = \overline{\mathfrak{A}}_1 + \cdots + \overline{\mathfrak{A}}_s.$$

The other half of the theorem is similarly proved.

3. *If the aggregate* $\mathfrak{A}$ *can be divided into the measurable unmixed aggregates* $\mathfrak{A}_1, \cdots \mathfrak{A}_s$, *it is measurable, and*
$$\text{Cont } \mathfrak{A} = \text{Cont } \mathfrak{A}_1 + \cdots + \text{Cont } \mathfrak{A}_s.$$

This follows as corollary of 2.

4. *Let* $\mathfrak{A}_1, \cdots \mathfrak{A}_s$ *be limited aggregates whose frontiers are discrete. Let* $\mathfrak{A}$ *be the union of these aggregates. Then* $\mathfrak{A}$ *is measurable, and*
$$\text{Cont } \mathfrak{A} = \text{Cont } \mathfrak{A}_1 + \cdots + \text{Cont } \mathfrak{A}_s.$$

For, we may divide $\mathfrak{A}$ into $\mathfrak{A}_1, \cdots \mathfrak{A}_s$, and these latter aggregates are unmixed, by hypothesis. The aggregates $\mathfrak{A}_1, \cdots \mathfrak{A}_s$ are also measurable by 706.

712. 1. Connected with any limited complete aggregate $\mathfrak{A}$ of upper content $\overline{\mathfrak{A}} > 0$ is an aggregate $\mathfrak{B}$, obtained from $\mathfrak{A}$ by a *process of sifting* as follows:

Let $D_1, D_2, \cdots$ be a set of rectangular divisions of space, each formed from the preceding, by superimposing a rectangular division on it. Let the norms of these divisions converge to 0.

The division D_1 effects a division of $\mathfrak{A}$ into unmixed partial aggregates. Let $\mathfrak{A}_1$ denote those partial aggregates whose upper content is > 0. Then, by 711, 2, $\overline{\mathfrak{A}}_1 = \overline{\mathfrak{A}}$.

Similarly, the division D_2 defines a partial aggregate of $\mathfrak{A}_1$ and hence of $\mathfrak{A}$, such that $\overline{\mathfrak{A}}_2 = \overline{\mathfrak{A}}$, etc. Let us consider the cells of D_n which contain points of $\mathfrak{A}_n$. As $n \doteq \infty$, these cells diminish in size, and in the limit define a set of points $\mathfrak{B}$. The upper content of the points of $\mathfrak{A}$ in the domain of any point of $\mathfrak{B}$ is > 0. Thus each point of $\mathfrak{B}$ is a limiting point of $\mathfrak{A}$, and hence a point of $\mathfrak{A}$. We shall prove, moreover, that $\mathfrak{B}$ is perfect.

For, suppose b were an isolated point of $\mathfrak{B}$. Let C be a cube whose center is b and whose volume is small at pleasure. Let $\mathfrak{a}$ be the points of $\mathfrak{A}$ in C. Let us divide C into smaller cubes, say of volume $\frac{1}{n}\bar{\mathfrak{a}}$. The points of $\mathfrak{A}$ in at least n of these new cells must have an upper content > 0. Thus there are other points of $\mathfrak{B}$ in C besides b. Hence $\mathfrak{B}$ has no isolated points. To show that $\mathfrak{B}$ is complete, let β be a limiting point of $\mathfrak{B}$; it is therefore a point of $\mathfrak{A}$. The upper content of the points of $\mathfrak{A}$ in any domain of β is > 0. β will therefore lie in one of the cells of D_n, $n = 1, 2, \cdots$. Hence it is a point of $\mathfrak{B}$.

Finally,

$$\overline{\mathfrak{A}} = \overline{\mathfrak{B}}.$$

For, any cell of D_n which contains a point of $\mathfrak{B}$ contains a point of $\mathfrak{A}_n$, and conversely any cell which contains a point of $\mathfrak{A}_n$ contains a point of $\mathfrak{B}$, or is at least adjacent to such a cell.

2. The aggregate $\mathfrak{B}$ may be called the *sifted aggregate* of $\mathfrak{A}$.

713. 1. We shall find it useful to extend the terms *cells*, *division of space into cells*, etc., as follows:

Let us suppose the points of any aggregate $\mathfrak{A}$, which may be $\mathfrak{R}_m$ itself, arranged in partial aggregates which we shall call *cells*, and which have the following properties:

1°. There are only a finite number of cells in a limited portion of space.

2°. The frontier of each cell is discrete.

3°. Each cell lies in a cube of side $\leqq \delta$.

4°. Points common to two or more cells must lie on the frontier of these cells.

We shall call this a *division of* $\mathfrak{A}$ *of norm* δ.

2. Let Δ be such a division of space. Let $\mathfrak{A}$ be a limited aggregate. As in 702, $\overline{\mathfrak{A}}_\Delta$ may denote the content of all the cells of Δ which contain at least one point of $\mathfrak{A}$; while $\underline{\mathfrak{A}}_\Delta$ may denote the content of those cells *all* of whose points lie in $\mathfrak{A}$.

3. *Let $\mathfrak{A}$ be an aggregate formed of certain of these cells, $\mathfrak{A}_1, \cdots \mathfrak{A}_s$. Then $\mathfrak{A}$ is measurable; and*

$$\mathrm{Cont}\,\mathfrak{A} = \mathrm{Cont}\,\mathfrak{A}_1 + \cdots + \mathrm{Cont}\,\mathfrak{A}_s.$$

This is a corollary of 711, 3.

714. *Let $\mathfrak{A}$ be a limited point aggregate, and Δ a division of space of norm δ, not necessarily a rectangular division. Then*

$$\lim_{\delta=0} \overline{\mathfrak{A}}_\Delta = \overline{\mathfrak{A}}, \qquad \lim_{\delta=0} \underline{\mathfrak{A}}_\Delta = \underline{\mathfrak{A}}. \tag{1}$$

Let us prove the first half of 1); the other half is similarly established.

For each $\epsilon > 0$ there exists a cubical division D of norm d, such that

$$\overline{\mathfrak{A}} \leq \overline{\mathfrak{A}}_D < \overline{\mathfrak{A}} + \epsilon/2. \tag{2}$$

Let D' be another cubical division of norm d'.

Let $\overline{\mathfrak{B}}_{D'}$ denote the volume of all those cubes containing points of $\overline{\mathfrak{A}}_D$. We can choose d' so small that

$$\overline{\mathfrak{A}}_D \leq \overline{\mathfrak{B}}_{D'} < \overline{\mathfrak{A}}_D + \epsilon/2.$$

Then 2) gives

$$\overline{\mathfrak{A}} \leq \overline{\mathfrak{B}}_{D'} < \overline{\mathfrak{A}} + \epsilon. \tag{3}$$

Let Δ be any division of space, not necessarily cubical, of norm $\delta < \frac{1}{2} d'$.

Then $\overline{\mathfrak{A}}_\Delta$ contains every point of $\mathfrak{A}$; and is a part of $\overline{\mathfrak{B}}_{D'}$, since the distance of $\mathfrak{A}$ to $\overline{\mathfrak{B}}_{D'}$ is $\leqq d'$. Hence, by 702, 3, and 713, 3,

$$\overline{\mathfrak{A}} \leq \overline{\mathfrak{A}}_\Delta \leq \overline{\mathfrak{B}}_{D'}.$$

This gives with 3)

$$\overline{\mathfrak{A}} \leq \overline{\mathfrak{A}}_\Delta < \overline{\mathfrak{A}} + \epsilon$$

for any $\delta \leq \frac{1}{2} d'$.

715. 1. Let $\mathfrak{A}$ be a limited aggregate. If $\mathfrak{A}$ is not *complete*, let us add to it its lacking limiting points. The resulting aggregate $\mathfrak{B}$ may be called *the completed aggregate of* $\mathfrak{A}$.

A limited aggregate $\mathfrak{A}$, and its completed aggregate $\mathfrak{B}$, have the same upper content.

For, let us effect a rectangular division D of norm d. The cells containing points of $\mathfrak{B}$ fall in two classes: 1°, those cells $d_1, d_2, \cdots$ containing points of $\mathfrak{A}$; 2°, those cells $e_1, e_2, \cdots$ containing no point of $\mathfrak{A}$. Each of these latter cells, as e_ι, is contiguous to at least one cell d_κ. If $e_\iota, \cdots$ are contiguous to d_κ, we will join them to d_κ, to form a new cell δ_κ, in such a way that each e-cell has been joined to some one d-cell.

The cells $\delta_1, \delta_2, \cdots$ together with the cells $d_1, d_2, \cdots$ which remain unchanged by this process of consolidation, define a division Δ of the kind considered in 713. The norm δ of this division is evanescent with d.

Now, for the division Δ,

$$\overline{\mathfrak{A}}_\Delta = \overline{\mathfrak{B}}_\Delta .$$

By 714, the left side $\doteq \overline{\mathfrak{A}}$. Hence

$$\mathfrak{B} = \overline{\mathfrak{A}}.$$

2. *The lower contents* $\underline{\mathfrak{A}}, \underline{\mathfrak{B}}$ *do not need to be equal.*

For example, let $\mathfrak{A}$ = rational points in the interval $J = (0, 1)$. Then $\mathfrak{B} = J$.

But

$$\underline{\mathfrak{A}} = 0, \qquad \underline{\mathfrak{B}} = 1.$$

3. *Let* $\mathfrak{A}$ *be measurable. Then* $\mathfrak{A}$, *and its completed aggregate* $\mathfrak{B}$, *have the same content.*

For, we have just seen that

$$\mathfrak{A} = \overline{\mathfrak{A}} = \overline{\mathfrak{B}}. \tag{1}$$

On the other hand, every inner point of $\mathfrak{A}$ is an inner point of $\mathfrak{B}$. Hence

$$\underline{\mathfrak{A}}_D \le \underline{\mathfrak{B}}_D .$$

Hence, passing to the limit,

$$\mathfrak{A} = \underline{\mathfrak{A}} \le \underline{\mathfrak{B}} \le \overline{\mathfrak{B}}. \tag{2}$$

Hence 1), 2) give

$$\mathfrak{A} = \underline{\mathfrak{B}} = \mathfrak{B}.$$

4. *If* $\mathfrak{A}$ *is measurable, the content of* $\mathfrak{A}$ *and its derivative* $\mathfrak{A}'$ *are equal.*

For, let $\mathfrak{B}$ be the completed aggregate of $\mathfrak{A}$. Since every inner point of $\mathfrak{A}$ is an inner point of $\mathfrak{A}'$, and every point of $\mathfrak{A}'$ is in $\mathfrak{B}$, we have for any cubical division D, of norm d,

$$\underline{\mathfrak{A}}_D \leq \underline{\mathfrak{A}}'_D \leq \overline{\mathfrak{A}}'_D \leq \overline{\mathfrak{B}}_D.$$

Passing to the limit $d=0$, this gives, since $\mathfrak{A}$ is measurable,

$$\mathfrak{A} \leq \underline{\mathfrak{A}}' \leq \overline{\mathfrak{A}}' \leq \overline{\mathfrak{B}}. \qquad (3$$

But, by 3, $\mathfrak{A} = \overline{\mathfrak{B}}$. Hence 3) gives

$$\mathfrak{A} = \underline{\mathfrak{A}}' = \overline{\mathfrak{A}}'.$$

716. *Let* $\mathfrak{A}$ *be a limited aggregate whose upper content is* $\overline{\mathfrak{A}}$. *Let* $\mathfrak{B}$ *be a partial aggregate depending on* u *such that*

$$\lim_{u=0} \overline{\mathfrak{B}} = \overline{\mathfrak{A}}.$$

Let D *be a rectangular division of norm* d. *Then for each* $\epsilon > 0$ *there exists a pair of numbers* u_0, d_0, *such that*

$$\overline{\mathfrak{A}}_D - \overline{\mathfrak{B}}_{u,D} < \epsilon \qquad (1$$

for any $0 < u \leq u_0$, $0 < d \leq d_0$.

For, if $d \leq d_0$,
$$\overline{\mathfrak{A}} \leq \overline{\mathfrak{A}}_D < \overline{\mathfrak{A}} + \epsilon/2;$$
and, if $u \leq u_0$,
$$\overline{\mathfrak{A}} - \epsilon/2 < \overline{\mathfrak{B}}_u.$$
But
$$\overline{\mathfrak{B}}_{u,D} \leq \overline{\mathfrak{A}}_D.$$
Thus
$$\overline{\mathfrak{A}} - \epsilon/2 < \overline{\mathfrak{B}}_u \leq \overline{\mathfrak{B}}_{u,D} \leq \overline{\mathfrak{A}}_D < \overline{\mathfrak{A}} + \epsilon/2,$$
which establishes 1).

Plane and Rectilinear Sections of an Aggregate

717. 1. Let $\mathfrak{A}$ be an aggregate in $\mathfrak{R}_m$. As $x = (x_1 \cdots x_m)$ ranges over $\mathfrak{A}$, x_ι will range over an aggregate $\mathfrak{x}_\iota$ on the x_ι-axis, which we call the *projection of* $\mathfrak{A}$ *on this axis.*

The points of $\mathfrak{R}_m$ for which one of the coördinates as x_ι has a fixed value $x_\iota = \xi_\iota$, lie in an $m-1$ way plane, which we shall say is *perpendicular to the* x_ι*-axis.* We may denote it by P_{ξ_ι} or more shortly, by P_ι. The points of $\mathfrak{A}$ in P_ι form a *plane section of* $\mathfrak{A}$ *corresponding to the point* ξ_ι in $\mathfrak{x}_\iota$, which we denote by $\mathfrak{P}_{\xi_\iota}$ or $\mathfrak{P}_\iota$. We also say $\mathfrak{P}_\iota$ is a *plane section of* $\mathfrak{A}$ *perpendicular to the* x_ι*-axis.*

2. As $x = (x_1 \cdots x_m)$ ranges over the points of $\mathfrak{A}$, the point $(x_1, \cdots x_{\iota-1}, 0, x_{\iota+1}, \cdots x_m)$ ranges over an aggregate $\mathfrak{X}_\iota$ in the plane $x_\iota = 0$, which may be called *the* $m-1$ *way plane* Π_ι *of the axes perpendicular to* x_ι. We call $\mathfrak{X}_\iota$, *the projection of* $\mathfrak{A}$ *on* Π_ι.

3. Let us fix all the coördinates of $x = (x_1 \cdots x_m)$, except x_ι. Then x describes a right line *parallel to the* x_ι*-axis.* Let $\mathfrak{a}_\iota$ denote the points of $\mathfrak{A}$ on one of these lines. We shall call it a *rectilinear section of* $\mathfrak{A}$, *parallel to the* x_ι*-axis.*

4. *Let* $\mathfrak{A}$ *be limited and complete. Then the* $\mathfrak{P}_\iota$ *and the* $\mathfrak{a}_\iota$, *also the* $\mathfrak{x}_\iota$ *and* $\mathfrak{X}_\iota$, *are complete.*

Let us show that the $\mathfrak{P}_\iota$ are complete. Let p be a limiting point in one of the $\mathfrak{P}_\iota$. Let

$$p_1, p_2 \cdots \tag{1}$$

be a sequence of points in this plane which $\doteq p$. Then 1) is a sequence in $\mathfrak{A}$, and as $\mathfrak{A}$ is complete, p lies in $\mathfrak{A}$, and hence in $\mathfrak{P}_\iota$.

Let us show that $\mathfrak{x}_\iota$ is complete. In fact, let q be one of its limiting points. Let $q_1, q_2 \cdots$ be a sequence in $\mathfrak{x}_\iota$ which $\doteq q$. In each plane section $\mathfrak{P}_{q_\kappa}$, take a point r_κ.

This gives a sequence

$$r_1, r_2 \cdots$$

whose limiting points lie in $\mathfrak{A}$, since $\mathfrak{A}$ is complete. Moreover, the projection of these limiting points is q.

718. 1. *Let* $\mathfrak{A}$ *be a measurable aggregate.*

Let $\mathfrak{x}_\sigma$ *denote those points of* $\mathfrak{x}_\iota$, *for which the upper content of the frontier points of* $\mathfrak{P}_\iota$ *is* $\geqq \sigma$. *Then* $\mathfrak{x}_\sigma$ *is discrete.*

For, let us effect a cubical division D of $\mathfrak{R}_m$ of norm d. This effects also a division of norm d of the x_ι-axis. Let $d_1, d_2 \cdots$ denote those intervals on this axis, embracing at least one point for which the frontier points of the corresponding plane section have upper content $\geqq \sigma$. If $\overline{\mathfrak{F}}_D$ denote the volume of those cells containing frontier points $\mathfrak{F}$ of $\mathfrak{A}$, we have

$$\overline{\mathfrak{F}}_D \geqq \sigma \Sigma d_\iota, \qquad \text{for any } D.$$

Let $d \doteq 0$. As $\mathfrak{A}$ is measurable,

$$\sigma \, \overline{\mathrm{Cont}} \, \mathfrak{x}_\sigma = 0.$$

As $\sigma > 0$,

$$\overline{\mathrm{Cont}} \, \mathfrak{x}_\sigma = 0.$$

2. In a similar manner we prove:

Let $\mathfrak{X}_\sigma$ denote those points of the projection of the measurable aggregate $\mathfrak{A}$ on the plane $x_\iota = 0$, for which the content of the frontier points on the corresponding rectilinear sections is $\geqq \sigma$. Then $\mathfrak{X}_\sigma$ is discrete.

3. *Let $\mathfrak{x}_\iota$ be the projection of the measurable aggregate $\mathfrak{A}$ on the x_ι-axis. Let D be a division of $\mathfrak{R}_m$ of norm d. Let $f_1, f_2 \cdots$ denote those intervals on the x_ι-axis containing frontier points of $\mathfrak{x}_\iota$. Let $\gamma > 0$, $\sigma > 0$ be taken small at pleasure. If $f'_1, f'_2 \cdots$ denote those f-intervals containing points of $\mathfrak{x}_\iota$, for which the upper content of the corresponding plane sections $\mathfrak{P}$ is $\geqq \gamma$, we can take d_0 so small that*

$$\Sigma f'_\iota < \sigma, \qquad d \leq d_0.$$

For, in the contrary case, the upper content $\overline{\mathfrak{F}}$ of the frontier points of $\mathfrak{A}$ is

$$\geqq \gamma\sigma. \tag{1}$$

But $\mathfrak{A}$ being measurable, $\mathfrak{F} = 0$, which contradicts 1).

Classes of Integrable Functions

719. 1. *Let $f(x_1 \cdots x_m)$ be continuous at the limiting points of the limited complete field $\mathfrak{A}$. Then f is integrable in $\mathfrak{A}$.*

For, reasoning similar to that of 352 shows that we can effect a cubical division D, such that the oscillation of f in each cell of D containing points of $\mathfrak{A}$ is $< \omega$. Then by 700, 4, f is integrable.

2. *In the limited complete aggregate $\mathfrak{A}$, let the limited function $f(x_1 \cdots x_m)$ be continuous, except at the points of a discrete aggregate $\mathfrak{B}$. Then f is integrable in $\mathfrak{A}$.*

Since $\mathfrak{B}$ is discrete, there exists a cubical division D such that the volume of those cells containing points of $\mathfrak{B}$ is $< \epsilon$.

Let $\mathfrak{C}_D$ denote those cells which contain points of $\mathfrak{A}$, but do not contain points of $\mathfrak{B}$. Since f is continuous in $\mathfrak{C}_D$, we can effect a cubical division D' of $\mathfrak{C}_D$, such that the oscillation of f in any cell of D' is $< \omega$.

Then by 700, 4, f is integrable in $\mathfrak{A}$.

3. *Let $f(x_1 \cdots x_m)$ have limited variation in the limited field $\mathfrak{A}$. Then f is integrable in $\mathfrak{A}$.*

For,
$$\Omega_D f = \Sigma \omega_\kappa d_\kappa \leq d^m \Sigma \omega_\kappa,$$
$$\leq d\tilde{\omega}, \qquad \text{by 709, 2).}$$

Hence
$$\lim \Omega_D f = 0,$$
and f is integrable, by 700, 2.

720. As in 504, 505, 507, and 508, we may establish the following theorems:

1. *Let $f(x_1 \cdots x_m)$ be a limited integrable function in the limited field $\mathfrak{A}$. Then $|f(x_1 \cdots x_m)|$ is integrable in $\mathfrak{A}$.* [507.]

2. *Let $f_1, f_2, \cdots f_r$ be limited integrable functions in the limited field $\mathfrak{A}$. If $c_1, c_2, \cdots c_r$ denote constants, then*
$$c_1 f_1 + c_2 f_2 + \cdots + c_r f_r \qquad \text{and} \qquad f_1 \cdot f_2 \cdots f_r$$
are integrable in $\mathfrak{A}$. [504, 505.]

3. The converse of 1 is not necessarily true. For example, in a rectangle R let
$$f(xy) = 1, \qquad \text{for } x, y \text{ rational};$$
$$= -1, \qquad \text{for other points in } R.$$

Obviously f is not integrable in R.

On the other hand, $|f|$ obviously is integrable.

4. The product $f \cdot g$ may be integrable without either f or g being integrable in $\mathfrak{A}$. For example, in a rectangle R let $f(xy)$ be defined as in 3; while
$$g(xy) = -1, \qquad \text{for } x, y \text{ rational};$$
$$= 1, \qquad \text{for other points of } R.$$

Then $fg = -1$ in R, and is hence integrable in R.

721. *Let $f(x_1 \cdots x_m)$ be integrable in the limited complete field $\mathfrak{A}$. Let $\mathfrak{C}$ be the points of $\mathfrak{A}$ at which f is continuous. Then $\overline{\mathfrak{C}} = \mathfrak{A}$.*

For, if $\mathfrak{A}$ is discrete, the theorem is true, even if f has no points of continuity in $\mathfrak{A}$. Let us therefore suppose $\overline{\mathfrak{A}} > 0$. Let $\mathfrak{B}$ be the partial aggregate formed from $\mathfrak{A}$ by the process of sifting, considered in 712.

Let D be a rectangular division, and d one of its cells containing points of $\mathfrak{B}$; we can choose D so that no cell has points of $\mathfrak{B}$ only on its sides. Let $\mathfrak{a}$ be the points of $\mathfrak{A}$ in d. Since $\mathfrak{a}$ is a partial aggregate of $\mathfrak{A}$, $f(x_1 \cdots x_m)$ is integrable in $\mathfrak{a}$. The reasoning of 508 shows now that f must be continuous at one point, at least, of $\mathfrak{a}$ and hence at an infinity of points of $\mathfrak{a}$.

Among these points, lie points of $\mathfrak{B}$. Thus every cell of the division D, which contains a point of $\mathfrak{B}$, contains a point of $\mathfrak{C}$.

Hence
$$\overline{\mathfrak{C}} = \overline{\mathfrak{A}}.$$

Generalized Definition of Multiple Integrals

722. *Let $f(x_1 \cdots x_m)$ be limited in the limited field $\mathfrak{A}$. Let Δ be any division of space of norm δ into cells $\delta_1, \delta_2, \cdots$, not necessarily rectangular. Let $\mathfrak{M}_\iota$, $\mathfrak{m}_\iota$ be respectively the maximum and minimum of f in δ_ι. Then*

$$\lim_{\delta=0} \overline{S}_\Delta = \lim_{\delta=0} \Sigma \mathfrak{M}_\iota \delta_\iota = \overline{\int_{\mathfrak{A}}} f d\mathfrak{A}, \qquad (1$$

$$\lim_{\delta=0} \underline{S}_\Delta = \lim_{\delta=0} \Sigma \mathfrak{m}_\iota \delta_\iota = \underline{\int_{\mathfrak{A}}} f d\mathfrak{A}. \qquad (2$$

Let D be a cubical division of norm d. Let $d_1, d_2, \cdots$ be the cells of D containing points of $\mathfrak{A}$. We may denote their volumes by the same letters. Let $M_\iota = \text{Max} f$, in d_ι; also $F \geqq \text{Max} |f|$ in $\mathfrak{A}$, and $\geqq 1$. Then for each $\epsilon > 0$, there exists a d such that

$$\left| \overline{S}_D - \overline{\int_{\mathfrak{A}}} \right| < \frac{\epsilon}{2}, \qquad (3$$

where, as usual,

$$\overline{S}_D = \Sigma M_\iota d_\iota.$$

Furthermore we may choose d so small that

$$\overline{\mathfrak{A}}_D - \overline{\mathfrak{A}} < \frac{\epsilon}{8\,F}, \tag{4}$$

where

$$\overline{\mathfrak{A}}_D = \Sigma d_\iota.$$

Consider now the division Δ. Those of its cells containing points of $\mathfrak{A}$ fall into two classes: 1°, those lying in only one cell of D; 2°, those lying in two or more cells of D. Let $\delta_{\iota 1}, \delta_{\iota 2}, \cdots$ be the cells of the 1° class lying in d_ι. Let $\delta'_1, \delta'_2, \cdots$ be all the cells of the 2° class. Then the content of all the cells of Δ containing points of $\mathfrak{A}$ is

$$\Sigma\delta_{\iota\kappa} + \Sigma\delta'_\iota = \overline{\mathfrak{A}}_\Delta.$$

But since the frontier of $\overline{\mathfrak{A}}_D$ is discrete, there exists a δ_0 such that

$$\Sigma\delta'_\iota < \frac{\epsilon}{16\,F}, \qquad \delta \le \delta_0. \tag{5}$$

As moreover $\overline{\mathfrak{A}}_\Delta \doteq \overline{\mathfrak{A}}$, by 714, we may suppose that

$$\overline{\mathfrak{A}}_\Delta - \overline{\mathfrak{A}} < \frac{\epsilon}{16\,F}, \qquad \delta \le \delta_0. \tag{6}$$

From 5), 6) we have

$$|\Sigma\delta_{\iota\kappa} - \overline{\mathfrak{A}}| < \frac{\epsilon}{8\,F}.$$

This with 4) gives finally

$$|\Sigma\delta_{\iota\kappa} - \overline{\mathfrak{A}}_D| < \frac{\epsilon}{4\,F}. \tag{7}$$

Now

$$\overline{S}_\Delta = \Sigma\mathfrak{M}_{\iota\kappa}\delta_{\iota\kappa} + \Sigma\mathfrak{M}'_\iota\delta'_\iota,$$

where $\mathfrak{M}_{\iota\kappa}$, $\mathfrak{M}'_\iota$ are the maxima of f in $\delta_{\iota\kappa}$, δ'_ι respectively. Hence

$$\overline{S}_\Delta \leqq \Sigma M_\iota \delta_{\iota\kappa} + F\Sigma\delta'_\iota, \tag{8}$$

since

$$\mathfrak{M}_{\iota\kappa} \leqq M_\iota, \qquad \mathfrak{M}'_\iota \le F.$$

Thus 5), 8) give

$$\overline{S}_\Delta < \Sigma M_\iota \delta_{\iota\kappa} + \frac{\epsilon}{4}. \tag{9}$$

Now

$$|S_D - \sum_{\iota\kappa} M_\iota \delta_{\iota\kappa}| = |\sum_\iota M_\iota (d_\iota - \sum_\kappa \delta_{\iota\kappa})|$$

$$\leq F|\sum_\iota d_\iota - \sum_{\iota\kappa} \delta_{\iota\kappa}| = F|\bar{\mathfrak{A}}_D - \Sigma\delta_{\iota\kappa}|$$

$$< \frac{\epsilon}{4}, \qquad \text{by 7)}. \tag{10}$$

Thus 9), 10) give

$$\bar{S}_\Delta < \bar{S}_D + \frac{\epsilon}{2}. \tag{11}$$

In the same way we may show that for a properly chosen cubical division E,

$$\bar{S}_E < \bar{S}_\Delta + \frac{\epsilon}{2}. \tag{12}$$

From 3), 11), 12) we have

$$\left|\bar{S}_\Delta - \overline{\int_{\mathfrak{A}}}\right| < \epsilon, \qquad \delta \leq \delta_0.$$

This proves 1). In a similar manner we may demonstrate 2).

723. *Let $f(x_1 \cdots x_m)$ be limited in the measurable field $\mathfrak{A}$. Let Δ be an unmixed division of $\mathfrak{A}$ of norm δ, into the cells $\delta_1, \delta_2, \cdots$ As usual let*

$$\underline{S}_\Delta = \Sigma m_\iota \delta_\iota, \qquad \bar{S}_\Delta = \Sigma M_\iota \delta_\iota.$$

Let $\mathfrak{m}$ be the maximum of $\underline{S}_\Delta$, and $\mathfrak{M}$ the minimum of $\bar{S}_\Delta$ for all divisions Δ, $\delta \doteq 0$. Then

$$\mathfrak{m} = \underline{\int_{\mathfrak{A}}} f d\mathfrak{A}, \qquad \mathfrak{M} = \overline{\int_{\mathfrak{A}}} f d\mathfrak{A}.$$

Let us divide one of the cells as δ_ι into two unmixed cells δ_ι', δ_ι''. This gives a new division Δ'. Then the term $m_\iota \delta_\iota$ in $\underline{S}_\Delta$, is replaced by the two terms

$$m_\iota' \delta_\iota' + m_\iota'' \delta_\iota'' \geq m_\iota \delta_\iota,$$

in $\underline{S}_{\Delta'}$. Hence

$$\underline{S}_\Delta \leq \underline{S}_{\Delta'}.$$

Similarly

$$\bar{S}_{\Delta'} \leq S_\Delta.$$

The theorem follows now from 722. For, there exists a division Δ, such that

$$\mathfrak{m} - \epsilon < \underline{S}_\Delta \leq \mathfrak{m}. \qquad (1$$

Let us now take a sequence of divisions $\Delta', \Delta'', \cdots$ whose norms $\doteq 0$; each $\Delta^{(n)}$ being formed by subdividing the cells of $\Delta^{(n-1)}$. Then

$$\underline{S}_\Delta \leqq \underline{S}_{\Delta'} \leqq \underline{S}_{\Delta''} \cdots \doteq \underline{\int}_{\mathfrak{A}} f d\mathfrak{A}. \qquad (2$$

From 1), 2), we have

$$\left| \underline{\int}_{\mathfrak{A}} f d\mathfrak{A} - \mathfrak{m} \right| < \epsilon ;$$

hence

$$\mathfrak{m} = \underline{\int}_{\mathfrak{A}} f d\mathfrak{A}, \qquad \text{etc.}$$

Properties of Integrals

724. *Let $f(x_1 \cdots x_m)$ be limited and integrable in the limited field $\mathfrak{A}$. Let $\mathfrak{B}$ be a partial aggregate depending on u, such that $\overline{\mathfrak{B}} \doteq \overline{\mathfrak{A}}$, as $u \doteq 0$. Then*

$$\lim_{u=0} \int_{\mathfrak{B}} f d\mathfrak{B} = \int_{\mathfrak{A}} f d\mathfrak{A}. \qquad (1$$

Since f is integrable over $\mathfrak{A}$,

$$\Omega_D f < \epsilon/2 \qquad (2$$

for any division D of norm $d \leq d_0$, by 700, 2.

Moreover, by 716, if d_0 and u_0 are taken small enough,

$$\overline{\mathfrak{A}}_D - \overline{\mathfrak{B}}_{u,D} < \epsilon/2F, \qquad (3$$

where $|f| < F$ in $\mathfrak{A}$.

Let $d_1, d_2, \cdots$ be the cells of D containing points of $\mathfrak{B}_u$, and $d_1', d_2', \cdots$ the cells containing only points of $\mathfrak{A}$. Then

$$\overline{S}_{\mathfrak{A}_D} = \Sigma M_\iota d_\iota + \Sigma M_\iota' d_\iota',$$

$$\overline{S}_{\mathfrak{B}_{u,D}} = \Sigma N_\iota d_\iota,$$

where $N_\iota = \text{Max } f$ for points of $\mathfrak{B}$ in d_ι.

Hence

$$|\overline{S}_{\mathfrak{A}_D} - \overline{S}_{\mathfrak{B}_{u,D}}| \leq \Sigma(M_\iota - N_\iota) d_\iota + F \Sigma d_\iota'$$

$$< \epsilon, \quad \text{by 2), 3)}. \qquad (4$$

Let now $d \doteq 0$. Then 4) gives

$$\left|\overline{\int}_{\mathfrak{A}} - \overline{\int}_{\mathfrak{B}_u}\right| \leq \epsilon.$$

Hence

$$\left|\int_{\mathfrak{A}} - \int_{\mathfrak{B}_u}\right| \leq \epsilon,$$

which gives 1).

725. 1. *Let* $f(x_1 \cdots x_m)$ *be limited in the measurable field* $\mathfrak{A}$. *Let* $\mathfrak{B}$ *be an outer field of* $\mathfrak{A}$. *Let* $g(x_1 \cdots x_m) = 0$ *in* $\mathfrak{B}$, *except at the points of* $\mathfrak{A}$, *where it* $= f(x_1 \cdots x_m)$. *Then*

$$\overline{\int}_{\mathfrak{A}} f d\mathfrak{A} = \overline{\int}_{\mathfrak{B}} g d\mathfrak{B}\, ; \quad \underline{\int}_{\mathfrak{A}} f d\mathfrak{A} = \underline{\int}_{\mathfrak{B}} g d\mathfrak{B}.$$

For, let D be a division of space of norm d, not necessarily rectangular. Let the inner cells of $\mathfrak{A}$ be $d_1, d_2, \cdots$ while $d'_1, d'_2, \cdots$ denote cells containing frontier points of $\mathfrak{A}$. Let M_ι, N_ι denote the maxima of f, g in d_ι, while M'_ι, N'_ι are the maxima of f, g in d'_ι. Then

$$\overline{S}_D = \Sigma M_\iota d_\iota + \Sigma M'_\iota d'_\iota,$$

$$\overline{T}_D = \Sigma N_\iota d_\iota + \Sigma N'_\iota d'_\iota.$$

Hence, since $M_\iota = N_\iota$, we have, setting $|f| \leq F$ in $\mathfrak{A}$,

$$|\overline{S}_D - \overline{T}_D| \leq F \Sigma d'_\iota.$$

Let now $d \doteq 0$. Since $\mathfrak{A}$ is measurable, we have the first part of our theorem. The second part follows likewise.

2. In a similar manner we establish the following theorems:

Let $f(x_1 \cdots x_m)$ *be limited in the measurable field* $\mathfrak{A}$. *Let* $g(x_1 \cdots x_m) = f$ *at inner points of* $\mathfrak{A}$, *and* $= 0$ *at frontier points. Then*

$$\underline{\int}_{\mathfrak{A}} g d\mathfrak{A} = \underline{\int}_{\mathfrak{A}} f d\mathfrak{A}\, ; \quad \overline{\int}_{\mathfrak{A}} g d\mathfrak{A} = \overline{\int}_{\mathfrak{A}} f d\mathfrak{A}.$$

3. *Let* $f(x_1 \cdots x_m)$, $g(x_1 \cdots x_m)$ *be limited in the measurable field* $\mathfrak{A}$. *Let them be equal except at the points of a discrete aggregate. Then*

$$\underline{\int}_{\mathfrak{A}} f d\mathfrak{A} = \underline{\int}_{\mathfrak{A}} g d\mathfrak{A}\, ; \quad \overline{\int}_{\mathfrak{A}} f d\mathfrak{A} = \overline{\int}_{\mathfrak{A}} g d\mathfrak{A}.$$

726. 1. *Let $f(x_1 \cdots x_m)$ be limited and integrable in the limited field $\mathfrak{A}$. Let $\mathfrak{B}$ be a partial aggregate of $\mathfrak{A}$, such that $\overline{\mathfrak{A}} = \overline{\mathfrak{B}}$. Then*

$$\int_{\mathfrak{A}} f d\mathfrak{A} = \int_{\mathfrak{B}} f d\mathfrak{B}. \qquad (1$$

Let D be a cubical division of space of norm δ. Let d'_ι denote those cells containing points of $\mathfrak{B}$, and d''_ι denote the cells containing points of $\mathfrak{A}$, but not of $\mathfrak{B}$. Then employing the usual notation,

$$\Sigma f(\xi_\iota) d_\iota = \Sigma f(\xi'_\iota) d'_\iota + \Sigma f(\xi''_\iota) d''_\iota.$$

Since $\Sigma d'' \doteq 0$ as $\delta \doteq 0$, we have 1) on passing to the limit.

2. *Let $f(x_1 \cdots x_m)$ be limited and integrable in the limited complete field $\mathfrak{A}$. Let $\mathfrak{C}$ denote the points of $\mathfrak{A}$ at which f is continuous. Then*

$$\int_{\mathfrak{A}} f d\mathfrak{A} = \int_{\mathfrak{C}} f d\mathfrak{C}.$$

This follows from 1 and 721.

727. *Let $f(x_1 \cdots x_m)$ be limited in the limited field $\mathfrak{A}$. If for any cubical divisions D, of norm $d \le d_0$,*

$$A \le \Sigma f(x_\iota) d_\iota \le B;$$

then

$$A \le \underline{\int_{\mathfrak{A}}} f d\mathfrak{A} \le \overline{\int_{\mathfrak{A}}} f d\mathfrak{A} \le B. \qquad (1$$

For, in each cell d_ι there are points x'_ι, x''_ι, such that

$$f(x'_\iota) < m_\iota + \sigma, \quad f(x''_\iota) > M_\iota - \sigma,$$

however small $\sigma > 0$ is chosen.

Hence

$$A \le \Sigma f(x'_\iota) d_\iota < \underline{S}_D + \sigma \Sigma d_\iota; \quad B \geqq \Sigma f(x''_\iota) d_\iota > \overline{S}_D - \sigma \Sigma d_\iota. \qquad (2$$

Let

$$\sigma < \frac{\epsilon}{\overline{\mathrm{Cont}}\ \mathfrak{A}}.$$

Then passing to the limit $d = 0$ in 2), we get

$$A - \epsilon \le \underline{\int_{\mathfrak{A}}} \le \overline{\int_{\mathfrak{A}}} \le B + \epsilon.$$

From this we conclude 1) at once.

728. *Let $f(x_1 \cdots x_m)$ be limited in the limited field $\mathfrak{A}$. Let $\mathfrak{A}$ be divided into the unmixed fields $\mathfrak{A}_1, \cdots \mathfrak{A}_s$. Then*

$$\overline{\int_{\mathfrak{A}}} f d\mathfrak{A} = \overline{\int_{\mathfrak{A}_1}} f d\mathfrak{A}_1 + \cdots + \overline{\int_{\mathfrak{A}_s}} f d\mathfrak{A}_s,$$

$$\underline{\int_{\mathfrak{A}}} f d\mathfrak{A} = \underline{\int_{\mathfrak{A}_1}} f d\mathfrak{A}_1 + \cdots + \underline{\int_{\mathfrak{A}_s}} f d\mathfrak{A}_s.$$

For, let $|f| \le F$ in $\mathfrak{A}$. Let D be a rectangular division of norm d. As in 711, 2, let $\mathfrak{A}_{1,D}, \cdots \mathfrak{A}_{s,D}$ be the cells containing points of $\mathfrak{A}_1, \cdots \mathfrak{A}_s$, respectively; while $\mathfrak{F}_D$ constitute the cells containing points of more than one of the fields $\mathfrak{A}_1 \cdots$ Then

$$|\overline{S}_D - (\overline{S}_{1,D} + \cdots + \overline{S}_{s,D})| \le sF\overline{\mathfrak{F}}_D.$$

Letting $d \doteq 0$ in this relation, we get the first part of the theorem. The rest is proved likewise.

729. As in 504; 489, 4; 526, 2; 531, we may prove the following theorems.

1. *Let $f_1, \cdots f_s$ be integrable in the limited field $\mathfrak{A}$. Let $c_1, \cdots c_s$ be constants and*

$$F = c_1 f_1 + \cdots + c_s f_s.$$

Then

$$\int_{\mathfrak{A}} F d\mathfrak{A} = c_1 \int_{\mathfrak{A}} f_1 d\mathfrak{A} + \cdots + c_s \int_{\mathfrak{A}} f_s d\mathfrak{A}. \qquad [504]$$

2. *Let $f(x_1 \cdots x_m)$ be integrable in the limited field $\mathfrak{A}$, and numerically $\le M$. Then*

$$\left| \int_{\mathfrak{A}} f d\mathfrak{A} \right| \le M\,\overline{\text{Cont}}\,\mathfrak{A}. \qquad [489, 4]$$

3. *Let $f(x_1 \cdots x_m)$, $g(x_1 \cdots x_m)$ be integrable in the limited field $\mathfrak{A}$, and let $f \le g$. Then*

$$\int_{\mathfrak{A}} f d\mathfrak{A} \le \int_{\mathfrak{A}} g a\mathfrak{A}. \qquad [526, 2]$$

4. *Let $f(x_1 \cdots x_m)$, $g(x_1 \cdots x_m)$ be integrable in the limited field $\mathfrak{A}$. Let*

$$f \geqq 0 ; \qquad \Theta = \text{Mean } g, \qquad \text{in } \mathfrak{A}.$$

Then

$$\int_{\mathfrak{A}} fg d\mathfrak{A} = \Theta \int_{\mathfrak{A}} f d\mathfrak{A}. \qquad [531]$$

5. The following theorems are readily proved:

Let $f(x_1 \cdots x_m)$ be limited in the limited field $\mathfrak{A}$. If $f(x_1 \cdots x_m) > \lambda$ in $\mathfrak{A}$,

$$\overline{\int}_{\mathfrak{A}} f d\mathfrak{A} \geqq \lambda \overline{\mathfrak{A}}.$$

6. *Let $f(x_1 \cdots x_m)$ be limited in the limited field $\mathfrak{A}$, and $\geqq 0$. If $\mathfrak{B}$ is a partial field of $\mathfrak{A}$,*

$$\overline{\int}_{\mathfrak{A}} f d\mathfrak{A} \geqq \overline{\int}_{\mathfrak{B}} f d\mathfrak{B}.$$

For,

$$\overline{S}_{\mathfrak{A}_D} \geqq \overline{S}_{\mathfrak{B}_D}.$$

7. *Let $f_1, \cdots f_n$ be limited in the limited field $\mathfrak{A}$. Then*

$$\overline{\int}_{\mathfrak{A}} (f_1 + \cdots + f_n) d\mathfrak{A} \leq \overline{\int}_{\mathfrak{A}} f_1 d\mathfrak{A} + \cdots + \overline{\int}_{\mathfrak{A}} f_n d\mathfrak{A},$$

$$\underline{\int}_{\mathfrak{A}} (f_1 + \cdots + f_n) d\mathfrak{A} \geq \underline{\int}_{\mathfrak{A}} f_1 d\mathfrak{A} + \cdots + \underline{\int}_{\mathfrak{A}} f_n d\mathfrak{A}.$$

For, in any cell d_ι of the division D,

$$\text{Max}(f_1 + \cdots + f_n) \leq \text{Max} f_1 + \cdots + \text{Max} f_n,$$

$$\text{Min}(f_1 + \cdots + f_n) \geqq \text{Min} f_1 + \cdots + \text{Min} f_n.$$

8. *Let f, g be limited in the limited field $\mathfrak{A}$. Then*

$$\overline{\int}_{\mathfrak{A}} (f - g) d\mathfrak{A} \geqq \overline{\int}_{\mathfrak{A}} f d\mathfrak{A} - \overline{\int}_{\mathfrak{A}} g d\mathfrak{A},$$

$$\underline{\int}_{\mathfrak{A}} (f - g) d\mathfrak{A} \leqq \underline{\int}_{\mathfrak{A}} f d\mathfrak{A} - \underline{\int}_{\mathfrak{A}} g d\mathfrak{A}.$$

For, in any cell d_ι of the division D,

$$\text{Max}(f-g) \geqq \text{Max}\, f - \text{Max}\, g,$$

$$\text{Min}(f-g) \leq \text{Min}\, f - \text{Min}\, g.$$

9. *Let f, g be limited in the limited field $\mathfrak{A}$, and let $f \leq g$. Then*

$$\underline{\int}_{\mathfrak{A}} fd\mathfrak{A} \leq \underline{\int}_{\mathfrak{A}} gd\mathfrak{A}\,; \quad \overline{\int}_{\mathfrak{A}} fd\mathfrak{A} \leq \overline{\int}_{\mathfrak{A}} fd\mathfrak{A}.$$

730. *Let $f(x_1 \cdots x_m)$ be limited in the limited field $\mathfrak{A}$. Let $\mathfrak{A}_\sigma$ denote the points of $\mathfrak{A}$ at which $|f| \geq \sigma$. If $\mathfrak{A}_\sigma$ is discrete for any $\sigma > 0$,*

$$\underline{\int}_{\mathfrak{A}} fd\mathfrak{A} = \overline{\int}_{\mathfrak{A}} fd\mathfrak{A} = 0.$$

For, let

$$\sigma < \epsilon/\overline{\mathfrak{A}}, \qquad |f| < F.$$

Let D be a division of norm δ. Let d'_κ denote the cells in which $|f| \geqq \sigma$, while d''_κ denotes the cells in which $|f| < \sigma$. Then

$$J_\delta = \Sigma f(\xi_\kappa) d_\kappa = \Sigma f(\xi'_\kappa) d'_\kappa + \Sigma f(\xi''_\kappa) d''_\kappa.$$

Hence

$$|J_\delta| < F \Sigma d'_\kappa + \sigma \Sigma d''_\kappa.$$

Let $\delta \doteq 0$. The right side is

$$\leq \sigma\, \overline{\text{Cont}}\, \mathfrak{A} < \epsilon,$$

which establishes the theorem, by 727.

731. *Let $f(x_1 \cdots x_m) \geqq 0$, be limited and integrable in the limited field $\mathfrak{A}$. If*

$$\int_{\mathfrak{A}} fd\mathfrak{A} = 0,$$

the points $\mathfrak{A}_\sigma$, at which $f(x) \geq \sigma$, an arbitrarily small positive number, form a discrete aggregate. Let $\mathfrak{Z}$ denote the points at which $f = 0$ If $\mathfrak{A}$ is complete,

$$\overline{\mathfrak{Z}} = \overline{\mathfrak{A}}.$$

For.

$$0 = \overline{\int}_{\mathfrak{A}} \geq \overline{\int}_{\mathfrak{A}_\sigma}, \qquad \text{by 729, 6.}$$

But

$$\overline{\int}_{\mathfrak{A}_\sigma} \geqq \sigma \overline{\mathfrak{A}}_\sigma, \qquad \text{by 729, 5.}$$

Hence

$$\sigma \overline{\mathfrak{A}}_\sigma = 0.$$

Since $\sigma > 0$,

$$\mathfrak{A}_\sigma \text{ must } = 0.$$

To prove the second part of the theorem, let b be a point of $\mathfrak{B}$, the sifted aggregate of $\mathfrak{A}$, at which f is continuous. Cf. 712.

Then if $f > 0$, we can choose $\delta > 0$ so small that $f \geq \lambda > 0$ in $V_\delta(b)$. But the upper content of the points $\mathfrak{a}$ of $\mathfrak{A}$ in V_δ is $\bar{\mathfrak{a}} > 0$. Hence

$$\int_{\mathfrak{A}} f d\mathfrak{A} \geqq \int_{\mathfrak{a}} f d\mathfrak{a} \geq \lambda \bar{\mathfrak{a}} > 0.$$

Hence $f = 0$ at every point of continuity of $\mathfrak{A}$ in $\mathfrak{B}$. Let now D be a rectangular division of space. The reasoning of 721 shows that every cell which contains a point of $\mathfrak{B}$ also contains a point of continuity lying in $\mathfrak{B}$. Hence,

$$\overline{\mathfrak{Z}}_D \geqq \overline{\mathfrak{B}}_D;$$

which gives,

$$\overline{\mathfrak{Z}} \geqq \overline{\mathfrak{B}}, \quad \text{or} \quad \overline{\mathfrak{Z}} = \overline{\mathfrak{B}}.$$

Reduction of Multiple Integrals to Iterated Integrals

732. 1. Let $f(x_1 \cdots x_m)$ be limited in the limited field $\mathfrak{A}$. Let $\mathfrak{x}_\iota$ be the projection of $\mathfrak{A}$ on the x_ι-axis. Let $\mathfrak{P}_\iota$ be a plane section of $\mathfrak{A}$ perpendicular to the x_ι-axis. Then the $(m-1)$tuple upper and lower integrals

$$\overline{\int}_{\mathfrak{P}_\iota} f d\mathfrak{P}_\iota, \qquad \underline{\int}_{\mathfrak{P}_\iota} f d\mathfrak{P}_\iota \tag{1}$$

are one-valued limited functions of x_ι, defined over $\mathfrak{x}_\iota$. For, let $\mathfrak{A}$ lie in a cube of side C. Let $|f(x_1 \cdots x_m)| \leq F$. Then both integrals are numerically

$$\leq FC^{m-1},$$

for any x_ι in $\mathfrak{x}_\iota$.

2. Each of the integrals 1), considered as functions of x_ι defined over $\mathfrak{x}_\iota$, have therefore upper and lower integrals in $\mathfrak{x}_\iota$, viz.:

$$\underline{\int}_{\mathfrak{x}_\iota} dx_\iota \underline{\int}_{\mathfrak{P}_\iota} fd\mathfrak{P}_\iota, \qquad \overline{\int}_{\mathfrak{x}_\iota} dx_\iota \underline{\int}_{\mathfrak{P}_\iota} fd\mathfrak{P}_\iota;$$

$$\underline{\int}_{\mathfrak{x}_\iota} dx_\iota \overline{\int}_{\mathfrak{P}_\iota} fd\mathfrak{P}_\iota, \qquad \overline{\int}_{\mathfrak{x}_\iota} dx \overline{\int}_{\mathfrak{P}_\iota} fd\mathfrak{P}_\iota.$$

For brevity these may also be written

$$\underline{\int}_{\mathfrak{x}_\iota} \underline{\int}_{\mathfrak{P}}, \quad \overline{\int}_{\mathfrak{x}_\iota} \underline{\int}_{\mathfrak{P}_\iota}, \quad \underline{\int}_{\mathfrak{x}_\iota} \overline{\int}_{\mathfrak{P}_\iota}, \quad \overline{\int}_{\mathfrak{x}_\iota} \overline{\int}_{\mathfrak{P}_\iota}.$$

733. 1. *Let $f(x_1 \cdots x_m)$ be limited in the measurable field $\mathfrak{A}$. Let $\mathfrak{x}_\iota$ be the projection of $\mathfrak{A}$ on the x_ι-axis. Let $\mathfrak{P}_\iota$ be the plane sections of $\mathfrak{A}$ corresponding to the points of $\mathfrak{x}_\iota$. Then*

$$\underline{\int}_{\mathfrak{A}} fd\mathfrak{A} \leq \underline{\int}_{\mathfrak{x}_\iota} dx_\iota \underline{\int}_{\mathfrak{P}_\iota} fd\mathfrak{P}_\iota \leq \overline{\int}_{\mathfrak{x}_\iota} dx_\iota \underline{\int}_{\mathfrak{P}_\iota} fd\mathfrak{P} \leq \overline{\int}_{\mathfrak{A}} fd\mathfrak{A}; \tag{1}$$

$$\underline{\int}_{\mathfrak{A}} fd\mathfrak{A} \leq \underline{\int}_{\mathfrak{x}_\iota} dx \overline{\int}_{\mathfrak{P}_\iota} fd\mathfrak{P}_\iota \leq \overline{\int}_{\mathfrak{x}_\iota} dx_\iota \overline{\int}_{\mathfrak{P}_\iota} fd\mathfrak{P}_\iota \leq \overline{\int}_{\mathfrak{A}} fd\mathfrak{A}. \tag{2}$$

Let us establish the relation 1); the demonstration of 2) is similar.

Let $\mathfrak{A}$ lie in an outer cube $\mathfrak{B}$, whose projection on the x_ι-axis is $\mathfrak{b}$, and whose plane sections perpendicular to this axis, we denote by $\mathfrak{Q}$.

We introduce an auxiliary function

$$\begin{aligned} g(x_1 \cdots x_m) &= f(x_1 \cdots x_m), \qquad \text{at points of } \mathfrak{A}; \\ &= 0, \qquad \text{at other points of } \mathfrak{B}. \end{aligned}$$

Let D be a cubical division of $\mathfrak{R}_m$ of norm d. This divides $\mathfrak{B}$ into cells which we denote by δ. It also divides the planes $\mathfrak{Q}$ into cells which we denote by δ'; and the segment $\mathfrak{b}$ into intervals which we denote by δ''.

Let M, M' denote the maxima; and m, m' the minima of $g(x_1 \cdots x_m)$ in the cells δ, δ'. Let $\overline{G}, \underline{G}$ be the upper and lower integrals of g in the field $\mathfrak{B}$. Let $|f(x_1 \cdots x_m)| \leq F$ in $\mathfrak{A}$. Then for each $\epsilon > 0$, there exists a d_0 such that

$$\underline{G} - \epsilon < \sum_{\mathfrak{B}} m\delta \leq \sum_{\mathfrak{B}} M\delta < \overline{G} + \epsilon, \qquad d \leq d_0. \qquad (3$$

Moreover, we note that for each x_ι of $\mathfrak{b}$,

$$\sum_{\mathfrak{Q}} m'\delta' \leq \int_{\mathfrak{Q}} g\,d\mathfrak{Q} \leq \sum_{\mathfrak{Q}} M'\delta'.$$

Or, since

$$m \leq m' \qquad M' \leq M,$$

$$\sum_{\mathfrak{Q}} m\delta' \leq \underline{\int}_{\mathfrak{Q}} g\,d\mathfrak{Q} \leq \sum_{\mathfrak{Q}} M\delta'.$$

Multiplying by δ'', and summing over $\mathfrak{b}$, we have, since $\delta = \delta' \; \delta''$,

$$\sum_{\mathfrak{B}} m\delta \leq \sum_{\mathfrak{b}} \delta'' \underline{\int}_{\mathfrak{Q}} \leq \sum_{\mathfrak{B}} M\delta.$$

Making use of 3), this gives

$$\underline{G} - \epsilon < \sum_{\mathfrak{b}} \delta'' \underline{\int}_{\mathfrak{Q}} < \overline{G} + \epsilon,$$

for any $d \leq d_0$. Thus, by 727,

$$\underline{G} - \epsilon < \underline{\int}_{\mathfrak{b}} \underline{\int}_{\mathfrak{Q}} \leq \overline{\int}_{\mathfrak{b}} \underline{\int}_{\mathfrak{Q}} \leq \overline{G} + \epsilon.$$

Or, since ϵ is small at pleasure,

$$\underline{G} \leq \underline{\int}_{\mathfrak{b}} \underline{\int}_{\mathfrak{Q}} \leq \overline{\int}_{\mathfrak{b}} \underline{\int}_{\mathfrak{Q}} \leq \overline{G}. \qquad (4$$

But, by 725,

$$\underline{G} = \underline{\int}_{\mathfrak{A}} f\,d\mathfrak{A}, \qquad \overline{G} = \overline{\int}_{\mathfrak{A}} f\,d\mathfrak{A}.$$

Thus

$$\underline{\int}_{\mathfrak{A}} f\,d\mathfrak{A} \leq \underline{\int}_{\mathfrak{b}} \underline{\int}_{\mathfrak{Q}} \leq \overline{\int}_{\mathfrak{b}} \underline{\int}_{\mathfrak{Q}} \leq \overline{\int}_{\mathfrak{A}} f\,d\mathfrak{A}. \qquad (5$$

Let $\overline{\mathfrak{F}}_{\mathfrak{P}}$ denote the upper content of the frontier points of $\mathfrak{P}_\iota$. Then

$$\left| \underline{\int}_{\mathfrak{P}_\iota} f\,d\mathfrak{P} - \underline{\int}_{\mathfrak{Q}} g\,d\mathfrak{Q} \right| \leq F\overline{\mathfrak{F}}_{\mathfrak{P}}. \qquad (6$$

Let

$$h(x_\iota)=\underline{\int}_{\mathfrak{P}_\iota} fd\mathfrak{P}_\iota, \qquad \text{for points of } \mathfrak{x}_\iota;$$

$$=0, \qquad \text{for other points of } \mathfrak{b}.$$

Then, by 718, 1; 730, and 6),

$$\underline{\int}_{\mathfrak{b}} h(x_\iota)dx_\iota=\underline{\int}_{\mathfrak{b}} dx_\iota \underline{\int}_{\mathfrak{Q}} gd\mathfrak{Q}.$$

But, by 718, 3,

$$\underline{\int}_{\mathfrak{b}} h(x_\iota)dx_\iota=\underline{\int}_{\mathfrak{x}_\iota} h(x_\iota)dx_\iota=\underline{\int}_{\mathfrak{x}_\iota} dx_\iota \underline{\int}_{\mathfrak{P}_\iota} fd\mathfrak{P}_\iota.$$

In the same way, we show

$$\overline{\int}_{\mathfrak{b}} dx_\iota \underline{\int}_{\mathfrak{Q}} gd\mathfrak{Q}=\overline{\int}_{\mathfrak{x}_\iota} dx_\iota \underline{\int}_{\mathfrak{P}_\iota} fd\mathfrak{P}_\iota.$$

Thus 1) has been proved.

2. In a similar manner we may demonstrate the following theorem.

Let $f(x_1 \cdots x_m)$ *be limited in the measurable field* $\mathfrak{A}$. *Let* $\mathfrak{X}_\iota$ *be the projection of* $\mathfrak{A}$ *on the plane* $x_\iota = 0$. *Let* $\mathfrak{a}_\iota$ *be the rectilinear sections of* $\mathfrak{A}$ *parallel to the* x_ι*-axis. Then*

$$\underline{\int}_{\mathfrak{A}} fd\mathfrak{A} \le \underline{\int}_{\mathfrak{X}_\iota} d\mathfrak{X}_\iota \underline{\int}_{\mathfrak{a}_\iota} fd\mathfrak{a}_\iota \le \overline{\int}_{\mathfrak{X}_\iota} d\mathfrak{X}_\iota \underline{\int}_{\mathfrak{a}_\iota} fd\mathfrak{a}_\iota \le \overline{\int}_{\mathfrak{A}} fd\mathfrak{A},$$

$$\underline{\int}_{\mathfrak{A}} fd\mathfrak{A} \le \underline{\int}_{\mathfrak{X}_\iota} d\mathfrak{X}_\iota \overline{\int}_{\mathfrak{a}_\iota} fd\mathfrak{a}_\iota \le \overline{\int}_{\mathfrak{X}_\iota} d\mathfrak{X}_\iota \overline{\int}_{\mathfrak{a}} fd\mathfrak{a}_\iota \le \overline{\int}_{\mathfrak{A}} fd\mathfrak{A}.$$

734. 1. As corollaries of 733 we have:

Let $f(x_1 \cdots x_m)$ *be integrable in the measurable field* $\mathfrak{A}$. *Then*

$$\int_{\mathfrak{A}} fd\mathfrak{A}=\int_{\mathfrak{x}_\iota} dx_\iota \underline{\int}_{\mathfrak{P}_\iota} fd\mathfrak{P}_\iota=\int_{\mathfrak{x}_\iota} dx_\iota \overline{\int}_{\mathfrak{P}_\iota} fd\mathfrak{P}_\iota$$

$$=\int_{\mathfrak{P}_\iota} d\mathfrak{P}_\iota \underline{\int}_{\mathfrak{x}_\iota} fdx_\iota=\int_{\mathfrak{P}_\iota} d\mathfrak{P}_\iota \overline{\int}_{\mathfrak{x}_\iota} fdx_\iota.$$

For, in this case,

$$\underline{\int}_{\mathfrak{A}}=\overline{\int}_{\mathfrak{A}}.$$

Hence the upper and lower integrals of $\underline{\int}_{\mathfrak{P}}$ and $\overline{\int}_{\mathfrak{P}}$ are equal.

2. *Let* $f(x_1 \cdots x_m)$ *be integrable in the measurable field* $\mathfrak{A}$, *as well as in each of its plane sections* $\mathfrak{P}$. *Then*

$$\int_{\mathfrak{A}} f d\mathfrak{A} = \int_{\mathfrak{a}} d\mathfrak{a} \int_{\mathfrak{P}} f d\mathfrak{P}.$$

If f *is integrable in each of the rectilinear sections* $\mathfrak{a}$, *we also have*

$$\int_{\mathfrak{A}} f d\mathfrak{A} = \int_{\mathfrak{P}} d\mathfrak{P} \int_{\mathfrak{a}} f dx.$$

EXAMPLE

735. Let us find the volume V of the ellipsoid E,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

By 708, 4, the surface forms a discrete aggregate; hence E has a volume. By 702, 2,

$$V = \int_E dx\, dy\, dz$$

$$= 8 \int_0^a dx \int_F dy\, dz,$$

where F is a quadrant of the ellipse

$$\frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 - \frac{x^2}{a^2}, \qquad 0 \leq x \leq a.$$

Hence

$$\int_F dy\, dz = \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} dy \int_0^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dz$$

$$= c \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} dy \sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}$$

$$= \tfrac{1}{4}\, bc\pi \left(1 - \frac{x^2}{a^2}\right).$$

Thus

$$V = 2\, \pi bc \int_0^a \left(1 - \frac{x^2}{a^2}\right) dx = \tfrac{4}{3}\, \pi abc.$$

736. *Let* $f(x_1 \cdots x_m)$ *be limited and integrable in the measurable field* $\mathfrak{A}$. *Let* $\sigma > 0$ *be arbitrarily small. Let* $\mathfrak{x}_\sigma$ *denote the points of* $\mathfrak{x}_\iota$ *for which*

$$\overline{\int}_{\mathfrak{P}_\iota} f d\mathfrak{P} - \underline{\int}_{\mathfrak{P}_\iota} f d\mathfrak{P} \geqq \sigma.$$

Then $\mathfrak{x}_\sigma$ *is discrete. If* $\mathfrak{A}$ *is complete, the upper content of the points where this difference vanishes is* $\bar{\mathfrak{x}}_\iota$. *A similar theorem holds for the differences*

$$\overline{\int}_{\mathfrak{a}_\iota} f dx_\iota - \underline{\int}_{\mathfrak{a}_\iota} f dx_\iota.$$

For, by 734, 1,

$$\int_{\mathfrak{A}} f d\mathfrak{A} = \int_{\mathfrak{a}}\overline{\int}_{\mathfrak{P}_\iota} = \int_{\mathfrak{a}}\underline{\int}_{\mathfrak{P}_\iota}.$$

Hence

$$\int_{\mathfrak{a}}\overline{\int}_{\mathfrak{P}} - \int_{\mathfrak{a}}\underline{\int}_{\mathfrak{P}} = \int_{\mathfrak{a}}\left\{\overline{\int}_{\mathfrak{P}} - \underline{\int}_{\mathfrak{P}}\right\} = 0.$$

The theorem follows now, by 731.

737. 1. *Let* $\mathfrak{A}$ *be a complete measurable field. Let* $\mathfrak{x}_\iota$ *be its projection on the* x_ι*-axis. Let* $\mathfrak{y}_\iota$ *be the points of* $\mathfrak{x}_\iota$ *for which the corresponding plane sections are measurable. Then* $\bar{\mathfrak{y}}_\iota = \bar{\mathfrak{x}}_\iota$.

A similar relation holds for the projection $\mathfrak{X}_\iota$.

For, let

$$f(x_1 \cdots x_m) = 1, \text{ at frontier points } \mathfrak{F} \text{ of } \mathfrak{A};$$
$$= 0, \text{ at other points of } \mathfrak{A}.$$

Then, by 702,

$$0 = \overline{\int}_{\mathfrak{F}} f d\mathfrak{F} = \underline{\int}_{\mathfrak{F}} f d\mathfrak{F}.$$

Since $\mathfrak{A}$ is measurable, $\mathfrak{F}$ is discrete, and hence measurable. Hence, by 734, 1,

$$0 = \int_{\mathfrak{x}_\iota} dx \overline{\int}_{\mathfrak{F}} f d\mathfrak{F}.$$

Hence, by 731, the points $\mathfrak{y}_\iota$ at which

$$\overline{\int}_{\mathfrak{F}} f d\mathfrak{F} = 0$$

have the same upper content as $\mathfrak{x}_\iota$.

2. *Let* $f(x_1 \cdots x_m)$ *be integrable in the measurable complete field* $\mathfrak{A}$. *Let* $\mathfrak{y}_\iota$ *denote those points of* $\mathfrak{x}_\iota$ *for which the integrals over the corresponding plane sections* $\mathfrak{P}_\iota$ *exist. Let* $\mathfrak{Y}_\iota$ *denote the points of* $\mathfrak{X}_\iota$ *for which the integrals over the corresponding rectilinear sections* $\mathfrak{a}_\iota$ *exist. Then*

$$\int_{\mathfrak{A}} f d\mathfrak{A} = \int_{\mathfrak{y}_\iota} dx_\iota \int_{\mathfrak{P}_\iota} f d\mathfrak{P}_\iota = \int_{\mathfrak{Y}_\iota} d\mathfrak{Y}_\iota \int_{\mathfrak{a}_\iota} f dx_\iota. \qquad (1$$

Let us prove the first half of 1); the other half follows similarly. By 734, 1

$$\int_{\mathfrak{A}} fd\mathfrak{A} = \int_{\mathfrak{X}_\iota} dx_\iota \underline{\int}_{\mathfrak{P}_\iota} fd\mathfrak{P}_\iota$$

$$= \int_{\mathfrak{Y}_\iota} dx_\iota \underline{\int}_{\mathfrak{P}_\iota} fd\mathfrak{P}_\iota, \qquad \text{by 726, 737,}$$

$$= \int_{\mathfrak{Y}_\iota} dx_\iota \int_{\mathfrak{P}_\iota} fd\mathfrak{P}_\iota,$$

since

$$\underline{\int}_{\mathfrak{P}_\iota} = \int_{\mathfrak{P}_\iota}$$

at the points $\mathfrak{y}_\iota$.

Application to Inversion

738. 1. In 570 we saw that

$$\int_a^\beta dy \int_a^b fdx$$

admits inversion, if $f(x, y)$ is limited in the rectangle $R = (ab\alpha\beta)$, and continuous except on a finite number of lines parallel to the x and y-axes. We can generalize this result as follows:

Let $f(x, y)$ be limited in the rectangle $R = (ab\alpha\beta)$. Let the points of discontinuity Δ in R be discrete. Let the points of Δ on any line parallel to either axis form a discrete aggregate on that line. Then

$$\int_a^\beta dy \int_a^b fdx, \quad \int_a^b dx \int_a^\beta fdy$$

exist and are equal.

For, by 719, 2, the double integral

$$\int_R fdR$$

exists. The theorem now follows from 734, 2.

2. If the points of discontinuity of $f(xy)$ on any line parallel to the x or y-axes do not form a discrete aggregate, we may apply the following theorem:

Let $f(xy)$ *be limited in the rectangle* $R = (ab\alpha\beta)$.

If the points of discontinuity of f *in* R *form a discrete aggregate, we have*

$$\int_\alpha^\beta dy \underline{\int_a^b} f dx = \int_\alpha^\beta dy \overline{\int_a^b} f dx = \int_a^b dx \underline{\int_\alpha^\beta} f dy = \int_a^b dx \overline{\int_\alpha^\beta} f dy.$$

739. 1. *Let* $f(x, y, z)$ *be limited in the rectangular parallelopiped* R, *bounded by the planes* $x = a$, $x = b$; $y = \alpha$, $y = \beta$; $z = A$, $z = B$. *Let it be continuous in* R, *except at the points of a discrete aggregate* Δ. *Let the points of* Δ *on any line or plane parallel to the axes be discrete with respect to that line or plane. Then the triple iterated integral*

$$\int_A^B dz \int_\alpha^\beta dy \int_a^b f(xyz) dx$$

exists, and admits unrestricted inversion.

For,

$$\int_R f(xyz) dR$$

exists by 719, 2. Let P denote a plane section of R parallel to the x, y-plane. Then the double integrals

$$\int_P f(xyz) dP, \qquad A \leq z \leq B,$$

also exist by 719, 2. Hence by 734, 2,

$$\int_R f dR = \int_A^B dz \int_P f dP.$$

But by 738,

$$\int_P f dP = \int_\alpha^\beta dy \int_a^b f dx = \int_a^b dx \int_\alpha^\beta f dy.$$

Hence,

$$\int_A^B \int_\alpha^\beta \int_a^b, \quad \int_A^B \int_a^b \int_\alpha^\beta$$

exist, and are equal. In the same way we may treat the four other inversions.

EXAMPLES

740. 1. Let $R = (0101)$. Let

$$f(x, y) = 0, \text{ for irrational } x, \text{ or } y;$$

$$= \frac{1}{n} \text{ for } x = \frac{m}{n};\ m, n \text{ relative prime, and } y \text{ rational.}$$

Then

$$\int_R f(xy)\,dR$$

exists by 701, Ex. 1. Its value is easily seen to be 0, by 730.

We have now:

$$\underline{\int} f dx = 0, \quad \underline{\int} f dy = 0.$$

Also

$$\overline{\int} f dx = 0, \text{ for irrational } y, \text{ obviously;}$$

$$= 0 \text{ for rational } y, \text{ by 730.}$$

Hence

$$\int_0^1 f dx = 0, \qquad \text{for any } y;$$

and therefore

$$\int_0^1 dy \int_0^1 f dx = 0.$$

On the other hand,

$$\overline{\int_0^1} f dy = \frac{1}{n}, \qquad \text{for } x = \frac{m}{n}.$$

Hence

$$\int_0^1 f dy$$

does not exist for rational x; *i.e.* for a point set which is not discrete.

However,

$$\underline{\int_0^1} f dy = 0$$

for any x. Hence, by 734, 1,

$$\int_R f dR = \int_0^1 dx \underline{\int_0^1} f dy;$$

which is obviously true.

This example illustrates the theorem of 736.

For, the points x at which

$$\overline{\int_0^1} f dy - \underline{\int_0^1} f dy \geqq \frac{1}{n},$$

are the rational points p/q whose denominators $q \leq n$.

Remark. Let R_σ denote the content of those points of R for which $f(xy) \geqq \sigma$

Obviously

$$R_\sigma = 0. \quad \text{Hence } \lim_{\sigma=0} R_\sigma = 0.$$

It would therefore be wrong to infer that

$$\text{Cont } \mathfrak{A} = 0,$$

because

$$\text{Cont } \mathfrak{A}_\sigma = 0,$$

for any $\sigma > 0$.

2. Let R be the rectangle (0101). Let us suppose the coördinates of its points x, y expressed in the dyadic system. Cf. 144. We define now a partial aggregate $\mathfrak{A}$ of R as follows. All its points lie on certain lines parallel to the y-axis, viz. $x = a$, where $0 < a < 1$ is any number having a finite representation. Let a particular value of a embrace p digits in its representation. Those points of the line $x = a$ belong to $\mathfrak{A}$, whose ordinate is expressed in p digits. Obviously this set of points is symmetrical with respect to x and y. If the representation of a is not finite, there is no point of $\mathfrak{A}$ on the line $x = a$, or on $y = a$. In any case, there are but a finite number of points of $\mathfrak{A}$ on any line parallel to the x or y-axis. Not so, for lines passing through a point of $\mathfrak{A}$, making an angle of 45° with the x-axis. Obviously, any little segment of such a line has an infinity of points of $\mathfrak{A}$ in it. Thus $\mathfrak{A}$ is dense.

Let us define now $f(xy)$ as follows:

$$f(xy) = 0, \text{ for any point of } \mathfrak{A};$$

$$= 1, \text{ for a point of } R \text{ not in } \mathfrak{A}.$$

Since the oscillation of f in any cell of R is 1, the double integral

$$\int_{\mathfrak{A}} f d\mathfrak{A}, \qquad \textit{does not exist.}$$

However,

$$\int_0^1 f dx = \int_0^1 f dy = 1.$$

Hence both iterated integrals

$$\int_0^1 dx \int_0^1 f dy, \qquad \int_0^1 dy \int_0^1 f dx$$

exist and are equal. (*Pringsheim.*)

3. In the rectangle $R = (0101)$ let us define another aggregate $\mathfrak{B}$ as follows. As before, the x of every point of $\mathfrak{B}$ must have a finite representation. If the representation of a embraces p digits, all the points of the line $x = a$ belong to $\mathfrak{B}$ whose ordinates are expressed by p or less digits.

Thus on any given line $x = a$, are only a finite number of points of $\mathfrak{B}$. On the contrary, on any little segment of the line $y = a$, lie an infinity of points of $\mathfrak{B}$. If the representation of a or b is not finite, there is no point of $\mathfrak{B}$ on the lines $x = a$, or $y = b$, as in Ex. 2.

Let us define $f(xy)$ as in Ex. 2.

$$f(xy) = 0, \text{ for any point in } \mathfrak{B};$$

$$= 1, \text{ for a point of } R \text{ not in } \mathfrak{B}.$$

Then as before, the double integral

$$\int_{\mathfrak{B}} f d\mathfrak{B} \qquad \text{does not exist.}$$

The integral

$$\int_0^1 fdx$$

does not exist for any y whose representation is finite, since

$$\underline{\int_0^1} fdx = 0, \qquad \overline{\int_0^1} fdx = 1.$$

On the other hand,

$$\int_0^1 fdy = 1, \qquad \text{for any } x.$$

Hence

$$\int_0^1 dx \int_0^1 fdy = 1. \qquad (\textit{Pringsheim.})$$

4. Let $f(xy)$ be limited in the rectangle $R = (0101)$. Let

$$\int_0^1 dy \int_0^1 f(xy)dx$$

exist. The reader might be tempted to conclude that therefore

$$\int_\alpha^\beta dy \int_a^b fdx, \qquad 0 \leq \alpha < \beta \leq 1, \qquad 0 \leq a < b \leq 1,$$

exists. To show this is not always so, let us set with *Du Bois Reymond*,

$$\begin{aligned} f(xy) &= 1, && \text{for rational } y; \\ &= 2x, && \text{for irrational } y. \end{aligned}$$

Then

$$\begin{aligned} \int_0^x fdx &= x, && \text{for rational } y; \\ &= x^2, && \text{for irrational } y. \end{aligned}$$

Hence

$$\int_0^1 fdx = 1, \qquad \text{for any } y;$$

and therefore

$$\int_0^1 dy \int_0^1 fdx = 1.$$

On the other hand,

$$\int_0^1 dy \int_0^b fdx, \qquad 0 < b < 1,$$

does not exist, since

$$\underline{\int_0^1} dy \int_0^b fdx = b^2, \qquad \overline{\int_0^1} dy \int_0^b fdx = b.$$

Transformation of the Variables

741. 1. Let

$$T; \qquad u_1 = \phi_1(t_1 \cdots t_m), \cdots \; u_m = \phi_m(t_1 \cdots t_m) \qquad (1$$

be defined over an aggregate $\mathfrak{A}$. These equations may be regarded as defining a transformation T, which transfers the points $t = (t_1 \cdots t_m)$ of $\mathfrak{A}$ to the points $u = (u_1 \cdots u_m)$. The points t, u

may be regarded as lying in the same space, or in different spaces. What we have called the image of $\mathfrak{A}$ defined by the equation 1) may now be regarded as the transformed aggregate of $\mathfrak{A}$, and may be denoted by $\mathfrak{A}_T$.

2. If the correspondence between the points of $\mathfrak{A}$ and $\mathfrak{A}_T$ is uniform, the equations 1) admit as solutions the one-valued inverse functions

$$t_1 = \theta_1(u_1 \cdots u_m), \cdots t_m = \theta_m(u_1 \cdots u_m). \qquad (2$$

The equations 2), regarded as a transformation, convert the points of $\mathfrak{A}_T$ back to $\mathfrak{A}$. For this reason it is called the *inverse transformation* of T, and denoted by T^{-1}.

3. The transformation

$$u_1 = t_1, \cdots u_m = t_m$$

is a special case of 1). As it leaves every point of $\mathfrak{A}$ essentially unaltered, it is called the *identical transformation*, and is denoted by 1.

4. Let

$$U; \qquad x_1 = \psi_1(u_1 \cdots u_m), \cdots x_m = \psi_m(u_1 \cdots u_m) \qquad (3$$

be another transformation defined over $\mathfrak{A}_T$, which we denote by U. The transformation resulting from the successive application of T and U is called their *product*, and is denoted by TU. The transformation which is applied first is written first.

The result of effecting T, and then its inverse, is to leave every point of $\mathfrak{A}$ at rest. Hence TT^{-1} is the identical transformation; in symbols

$$TT^{-1} = 1.$$

5. If $\mathfrak{A}_T$ goes over into $\mathfrak{A}_{TU}$ on applying U to $\mathfrak{A}_T$, we may regard $\mathfrak{A}_{TU}$ as the image of $\mathfrak{A}$, afforded by the equations 1), when we consider the ϕ's as functions of the x's through the u's, as given by 3).

6. The functions ϕ, ψ being one-valued, to any point in $\mathfrak{A}$ corresponds one point in $\mathfrak{A}_T$, $\mathfrak{A}_{TU}$. Suppose the correspondence between $\mathfrak{A}$, $\mathfrak{A}_{TU}$ is uniform. Then the correspondence between $\mathfrak{A}$, $\mathfrak{A}_T$ and between $\mathfrak{A}_T$, $\mathfrak{A}_{TU}$ is uniform.

For, suppose, for example, that to the point u of $\mathfrak{A}_T$ correspond two points t, t' of $\mathfrak{A}$. If now x corresponds to u; to the point x will correspond at least the two points t, t'. The correspondence is thus not uniform between $\mathfrak{A}, \mathfrak{A}_{TU}$.

7. If the functions ϕ have first partial derivatives in $\mathfrak{A}$, we call

$$J_T = \frac{\partial(\phi_1 \cdots \phi_m)}{\partial(t_1 \cdots t_m)} = \begin{vmatrix} \frac{\partial \phi_1}{\partial t_1} \cdots \frac{\partial \phi_m}{\partial t_1} \\ \cdot \quad \cdot \quad \cdot \quad \cdot \\ \frac{\partial \phi_1}{\partial t_m} \cdots \frac{\partial \phi_m}{\partial t_m} \end{vmatrix}$$

the *determinant of the transformation.*

If the first partial derivatives of the ϕ's are continuous in a region R, while the first partial derivatives of the ψ's are continuous in a region containing R_T, we have, by direct multiplication of the determinants J_T, J_U, and using 430, 5),

$$J_{TU} = J_T \cdot J_U, \tag{4}$$

which we may state roughly thus:

The determinant of the product of two transformations is the product of their determinants.

742. 1. Let

$$T; \qquad u_1 = \phi_1(t_1 \cdots t_m), \cdots u_m = \phi_m(t_1 \cdots t_m) \tag{1}$$

have continuous first partial derivatives in the region R. Let the correspondence between R and R_T be uniform. Let the determinant of the transformation $J_T \neq 0$ in R. In this case we shall say the transformation T defined by the equations 1) is *regular in R.*

2. *Let T be a regular transformation in R. Let t be a point of R, to which corresponds the point u. Let E be the image of $D_\delta(t)$. There exists an $\eta > 0$, such that $D_\eta(u)$ lies in E. Furthermore, if t runs over an inner aggregate $\mathfrak{A}$ of R, the η's do not sink below some positive number η_0.*

For, suppose there exists no $\eta > 0$, such that $D_\eta(u)$ lies in E. Then there exists a sequence of points $u_1, u_2, \cdots$ which $\doteq u$, and which do not lie in E. The inverse functions of 1) being one-valued and continuous about u, by 443, the image of the above points form a sequence $t_1, t_2, \cdots$ which $\doteq t$. Hence all the t_n for $n \geqq$ some m lie in $D_\delta(t)$, and thus $u_m, u_{m+1}, \cdots$ must lie in E, which is a contradiction. This establishes the first part of the theorem. Turning to the second part, suppose $\eta \doteq 0$ as t runs over $\mathfrak{A}$. Then reasoning similar to that of 352 leads at once to a contradiction.

3. *Let T be a regular transformation in the region R. Then R_T is a region. Let $\mathfrak{A}$ be an inner aggregate of R. To inner and frontier points of $\mathfrak{A}$, correspond respectively inner and frontier points of $\mathfrak{B} = \mathfrak{A}_T$, and conversely.*

This is a direct consequence of 2.

4. *If T is a regular transformation in the region R, T^{-1} is a regular transformation in R_T. The determinant of the inverse transformation is*

$$J_{T^{-1}} = \frac{1}{J_T}.$$

This follows at once from 443 and 741, 4).

5. *Let T be a regular transformation in the region R. Let $\mathfrak{A}$ be an inner aggregate of R, and let $\mathfrak{B}$ be its image. If either $\mathfrak{A}$ or $\mathfrak{B}$ is measurable, the other is. If one is discrete, the other is.*

This follows from 4 and 708, 3.

743. 1. Let T be a regular transformation

$$T; \qquad x_1 = \phi_1(t_1 \cdots t_m), \cdots x_m = \phi_m(t_1 \cdots t_m) \tag{1}$$

in the region R. Since $J_T \neq 0$, not all the derivatives

$$\frac{\partial \phi_\iota}{\partial t_\kappa}$$

vanish at any point of R. To fix the ideas, let

$$\frac{\partial \phi_m}{\partial t_m} \neq 0 \tag{2}$$

at a point t, and hence, as it is continuous, in a certain domain $D_\delta(t)$ of t.

We show now how T can be expressed as the product of two special regular transformations. The first transformation we define thus:

$$T_1; \qquad u_1 = t_1, \cdots u_{m-1} = t_{m-1}, u_m = \phi_m(t_1 \cdots t_m). \tag{3}$$

By virtue of 2) this system may be inverted, giving

$$t_1 = u_1, \cdots t_{m-1} = u_{m-1}, t_m = \theta(u_1 \cdots u_m).$$

Here θ is one-valued, and has continuous first partial derivatives in a certain domain $D_\eta(u)$. If $\delta' \leq \delta$ is taken sufficiently small, the image U of $D_{\delta'}(t)$ lies in $D_\eta(u)$.

We define the transformation T_2 over U by

$$T_2; \quad x_1 = \phi_1(u_1 \cdots u_{m-1}\theta), \cdots x_{m-1} = \phi_{m-1}(u_1 \cdots u_{m-1}\theta), x_m = u_m,$$

where θ is the above one-valued function of the u's.

We see at once that

$$T = T_1 \cdot T_2,$$

when t ranges over $D_{\delta'}(t)$.

Since

$$J_{T_1} = \frac{\partial \phi_m}{\partial t_m} \neq 0,$$

and

$$J_T = J_{T_1} J_{T_2} \neq 0,$$

it follows that $J_{T_2} \neq 0$ in U. Since the correspondence is uniform, and the functions ϕ, ψ have continuous first derivatives in the respective domains, the two transformations T_1, T_2 are regular.

2. *Let $\mathfrak{A}$ be a limited inner aggregate of the region R. We can effect a cubical division of the t space of norm d such that for the points of $\mathfrak{A}$ in each cell d_κ, there exist two transformations $T_1^{(\kappa)}$, $T_2^{(\kappa)}$ of the type just considered, such that*

$$T = T_1^{(\kappa)} T_2^{(\kappa)}. \tag{4}$$

For, we can take d so small that not all the first partial derivatives vanish in any cell. For if they did, reasoning similar to that of 264 shows that they must then vanish at some point of R, which would require $J = 0$ at that point. Thus these cubes may be taken as the domains $D_\delta(t)$ in 1. By reasoning similar to that of 352, we show that the norms η of the domains $D_\eta(u)$ considered in 1 do not sink below some positive number. The same reasoning applied to the norms δ' of the $D_{\delta'}(t)$ above, shows that the norms δ' of the above $D_{\delta'}(t)$, are all greater than some positive number. Thus if d is taken small enough, the relation 4) will hold in each cell containing points of $\mathfrak{A}$.

744. *Let*

T; $$x_1 = \phi_1(t_1 \cdots t_m), \cdots x_m = \phi_m(t_1 \cdots t_m)$$

define a regular transformation of determinant J in the region R. Let $\mathfrak{T}$ be any inner measurable perfect aggregate of R, and let $\mathfrak{X}$ be its image. Let $f(x_1 \cdots x_m)$ be continuous in $\mathfrak{X}$. Then

$$\int_{\mathfrak{X}} f(x_1 \cdots x_m) dx_1 \cdots dx_m = \int_{\mathfrak{T}} |J| f dt_1 \cdots dt_m. \tag{1}$$

For $m = 1$ the relation 1) is easily seen to be true, taking account of direction in $\mathfrak{T}$. Let us therefore assume it is correct for $m - 1$, and show it is so for m. Let D be a cubical division of the t space, such that in each cell containing points of $\mathfrak{T}$ the transformation T can be expressed as the product of two transformations

T_1; $\quad u_1 = t_1, \cdots u_{m-1} = t_{m-1}, u_m = \phi_m(t_1 \cdots t_m)$;

T_2; $\quad x_1 = \phi_1(u_1 \cdots u_{m-1}\theta), \cdots x_{m-1} = \phi_{m-1}(u_1 \cdots u_{m-1}\theta), x_m = u_m,$

of the type considered in 743.

Let J_1, J_2 be their determinants. Then

$$J = J_1 J_2.$$

If the relation 1) holds for each of the partial aggregates into which $\mathfrak{T}$ falls after effecting D, it obviously holds in $\mathfrak{T}$, by 728. We may therefore assume, without loss of generality, that the same transformations T_1, T_2 may be employed throughout $\mathfrak{X}$.

Let the image of $\mathfrak{X}$ in the u space be $\mathfrak{U}$. We have now

$$\int_{\mathfrak{X}} f dx_1 \cdots dx_m = \int_{\mathfrak{y}_m} dx_m \int_{\mathfrak{P}_m} f dx_1 \cdots dx_{m-1}, \qquad \text{by 737, 2,}$$

$$= \int_{\mathfrak{y}'_m} du_m \int_{\mathfrak{P}'_m} |J_2| f du_1 \cdots du_{m-1}, \qquad \text{by hypothesis,}$$

where $\mathfrak{y}'_m$, $\mathfrak{P}'_m$ are the transformed $\mathfrak{y}_m$, $\mathfrak{P}_m$, respectively.

Since $\mathfrak{X}$ is measurable, $\mathfrak{U}$ is so, by 742, 5. The same is true of $\mathfrak{P}'_\iota$. Hence, by 737, 2,

$$\int_{\mathfrak{X}} f dx_1 \cdots dx_m = \int_{\mathfrak{U}} |J_2| f du_1 \cdots du_m.$$

Applying the transformation T_1 to the integral on the right, similar considerations show that

$$\int_{\mathfrak{X}} f dx_1 \cdots dx_m = \int_{\mathfrak{T}} |J_1||J_2| f dt_1 \cdots dt_m,$$

which is 1).

745. *Let*

$$x_1 = \phi_1(t_1 \cdots t_m), \quad \cdots x_m = \phi_m(t_1 \cdots t_m)$$

define a regular transformation of determinant J, in the region R. To a rectangular division D of norm d of the t-space into cells d_κ, corresponds a division Δ of norm δ of the x-space into cells δ_κ. Let $\mathfrak{T}$ be any inner region of R, and $\mathfrak{X}$ its image. The cells of Δ falling within $\mathfrak{X}$ are unmixed, and their contents are

$$\delta_\kappa = |J| d_\kappa + \epsilon_\kappa d_\kappa, \qquad t \text{ in } d_\kappa \tag{1}$$

where $|\epsilon_\kappa| < \epsilon$ uniformly, on taking d sufficiently small.

For, $\mathfrak{T}$ being an inner region, to each inner rectangular cell d_κ of $\mathfrak{T}$, corresponds a measurable cell δ_κ of $\mathfrak{X}$ by 742, 5. Hence the cells δ_κ within $\mathfrak{X}$ are unmixed, limited, perfect, and finite in number for any Δ.

Since the determinant J is continuous in $\mathfrak{T}$, we can take d_0 so small that in *any* d_κ in $\mathfrak{T}$,

$$|J| = |J_{\tau_\kappa}| + \sigma_\kappa; \tag{2}$$

where τ_κ is any point d_κ, and

$$|\sigma_\kappa| < \epsilon,$$

for any division D of norm $\leq d_0$.

From 702, 2 and 744, 1), we have for divisions of norm $\leq d_0$,

$$\begin{aligned} \text{Cont } \delta_\kappa = \int_{\delta_\kappa} dx_1 \cdots dx_m &= \int_{d_\kappa} |J| dt_1 \cdots dt_m \\ &= |J_{\tau_\kappa}| \int_{d_\kappa} dt_1 \cdots dt_m + \int_{d_\kappa} \sigma_\kappa dt_1 \cdots dt_m, \text{ by (2,} \\ &= |J_{\tau_\kappa}| d_\kappa + \epsilon_\kappa d_\kappa, \end{aligned}$$

where

$$|\epsilon_\kappa| < \epsilon.$$

746. 1. *Let*

$$T; \qquad x_1 = \phi_1(t_1 \cdots t_m), \cdots x_m = \phi_m(t_1 \cdots t_m)$$

define a regular transformation of determinant J, *in a region* R. *Let* $\mathfrak{T}$ *be any inner aggregate of* R, *and let* $\mathfrak{X}$ *be its image. Let* $f(x_1 \cdots x_m)$ *be limited in* $\mathfrak{X}$. *Then*

$$\overline{\int}_{\mathfrak{X}} f(x_1 \cdots x_m) d\mathfrak{X} = \overline{\int}_{\mathfrak{T}} f|J|\, d\mathfrak{T}, \qquad (1$$

$$\underline{\int}_{\mathfrak{X}} f(x_1 \cdots x_m) d\mathfrak{X} = \underline{\int}_{\mathfrak{T}} f|J|\, d\mathfrak{T}. \qquad (2$$

For, let us effect a cubical division of the t-space of norm d. To it corresponds a division of $\mathfrak{X}$ into cells of norm δ.

Let us consider the integral on the left of 1). Using the customary notation, we have, letting J_ι denote the value of J at some point in d_ι,

$$\Sigma M_\iota \delta_\iota = \Sigma M_\iota(|J_\iota| + \epsilon_\iota) d_\iota, \qquad \text{by 745, 1),}$$

$$= \Sigma M_\iota |J_\iota| d_\iota + \Sigma \epsilon_\iota M_\iota d_\iota. \qquad (3$$

But if $|f| < F$ in $\mathfrak{X}$,

$$|\Sigma \epsilon_\iota M_\iota d_\iota| < \epsilon F(\overline{\text{Cont}}\ \mathfrak{T} + \epsilon) = \eta. \qquad (4$$

Let us now consider the integral on the right of 1). In the cell d_ι,

$$\text{Max}\, f \cdot \text{Min}\, |J| \leq \text{Max} \cdot f|J| \leq \text{Max}\, f \cdot \text{Max}\, |J|,$$

if Max f is positive; while the signs are reversed, if it is negative. Let us set

$$M_\iota' = \text{Max} \cdot f|J|, \qquad \text{in } d_\iota.$$

Then

$$M_\iota' = M_\iota(|J_\iota| + \epsilon_\iota'), \qquad |\epsilon_\iota'| < \epsilon \text{ uniformly,}$$

since the oscillation of J in any d_ι is uniformly $< \epsilon$. Thus

$$\Sigma M_\iota' d_\iota = \Sigma M_\iota |J_\iota| d_\iota + \Sigma \epsilon_\iota' M_\iota d_\iota, \qquad (5$$

where, as in 4),

$$|\Sigma \epsilon_\iota' M_\iota d_\iota| < \eta. \qquad (6$$

Thus 3), 5) give in connection with 4), 6),

$$|\Sigma M_\iota \delta_\iota - \Sigma M_\iota' d_\iota| < 2\,\eta,$$

which establishes the relation 1). Similarly we may prove 2).

2. From 1 a great variety of theorems may be deduced by a passage to the limit. We note here only the following:

Let

T; $$x_1 = \phi_1(t_1 \cdots t_m), \cdots x_m = \phi_m(t_1 \cdots t_m)$$

be continuous in the measurable aggregate $\mathfrak{T}$, *containing all its frontier points* $\mathfrak{F}$; *and regular in any inner measurable aggregate* $\mathfrak{U}$. *Let the determinant of* T *be limited in* $\mathfrak{T}$. *Let* $\mathfrak{X}$, *the image of* $\mathfrak{T}$, *be measurable also; and let its frontier be the image of* $\mathfrak{F}$. *Let* $f(x_1 \cdots x_m)$ *be limited in* $\mathfrak{X}$. *Then*

$$\int_{\mathfrak{X}} f(x_1 \cdots x_m) d\mathfrak{X} = \int_{\mathfrak{T}} f|J| \, d\mathfrak{T},$$

provided either integral exists.

For, let $\mathfrak{Y}$ be the image of $\mathfrak{U}$. Then, by **1**,

$$\int_{\mathfrak{Y}} = \int_{\mathfrak{U}}.$$

Let now $\mathfrak{U} \doteq \mathfrak{T}$, then $\mathfrak{Y} \doteq \mathfrak{X}$.

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